

SOLITON SOLUTIONS OF THE KP EQUATION AND APPLICATION TO SHALLOW WATER WAVES

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ABSTRACT. The main purpose of this paper is to give a survey of recent developments on a classification of soliton solutions of the KP equation. The paper is self-contained, and we give complete proofs of theorems needed for the classification. The classification is based on the totally non-negative cells in the Schubert decomposition of the real Grassmann manifold, $\text{Gr}(N, M)$, the set of N -dimensional subspaces in \mathbb{R}^M . Each soliton solution defined on $\text{Gr}(N, M)$ asymptotically consists of the N number of line-solitons for $y \gg 0$ and the $M - N$ number of line-solitons for $y \ll 0$. In particular, we give detailed description of the soliton solutions associated with $\text{Gr}(2, 4)$, which play a fundamental role in the study of multi-soliton solutions. We then consider a physical application of some of those solutions related to the Mach reflection discussed by J. Miles in 1977.

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1. INTRODUCTION

The purpose of this paper is to give a survey of our recent works on a classification theory of soliton solutions of the Kadomtsev-Petviashvili (KP) equation [4, 17, 3, 6, 7]. We also explain the detailed structure of some of the soliton solutions obtained in this classification theory, and discuss a physical application of a subclass of those solutions, which is related to the resonant interactions of the solitary waves in shallow water. Most of the materials in this paper are based on a series of lectures given by one of the authors (YK) at Chinese Academy of Science in Beijing after the international conference “Nonlinear Waves - Theory and Applications”, on June 9-12, 2008. It is a pleasure to thank Professors Qing-Ping Liu, Xing-Biao Hu and Ke Wu for the arrangement of the lectures, and Mr. Kai Tian for making lecture notes.

In 1970, Kadomtsev and Petviashvili [15] proposed a two-dimensional dispersive wave equation to study the stability of one soliton solution of the KdV equation under the influence of weak transverse perturbations. This equation is now referred to as the KP equation, and considered to be a prototype of the integrable nonlinear dispersive wave equations in two-dimension. The KP equation is a completely integrable system with remarkably rich mathematical structure which is well documented in several monographs (see for examples, [1, 2, 8, 14, 20, 24, 25, 28]). These mathematical structures include the existence of N -soliton solutions, the Lax formulation of the inverse scattering transform, and the existence of an infinite dimensional symmetries. However the real breakthrough in the KP theory occurred in 1981 by Sato [31] who realized that the solution of the KP equation is given by a $GL(\infty)$ -orbit on an infinite dimensional Grassmann manifold (Sato universal Grassmannian), and the KP equation is just a Plücker relation on the Grassmannian. The present paper deals with a finite dimensional version of the Sato theory, and in particular, we are interested in the solutions which are real and non-singular in the entire xy -plane where they are localized along certain rays.

It is quite important to recognize that the resonant interaction plays a fundamental role in multi-dimensional waves. The original description of the soliton interaction of the KP equation was based on a two-soliton solution found in the Hirota form, which has the shape of “X” describing an intersection of two lines with some angle and a shift of lines at the intersection (the phase shift). In 1977, Miles [22] pointed out that this two-soliton solution (referred to as the “O-type” solution, where “O” stands for *original*) becomes singular if the angle of the intersection is smaller than certain critical value. Since the KP equation was proposed to describe *quasi*-two dimensional waves, it should admit solutions that provide good approximations for two soliton interaction with smaller angle. Thus, it seems strange that one does not have a reasonable solution in the parameter regimes where the KP equation is supposed to give a better approximation. Miles also found that at the critical angle the two line-solitons of the O-type solution interact resonantly, and a third wave (soliton) is created to make a “Y-shape” solution. Indeed, it turns out that such Y-shape resonant solutions are exact solutions of the KP equation (see also [26]).

After the discovery of the resonant phenomena in the KP equation, several numerical and experimental studies were performed to investigate resonant interactions in other *real* two dimensional equations such as the ion-acoustic and shallow water wave equations under the Boussinesq approximation (see for examples [18, 19, 27, 10, 23]). However, after these activities, no significant progress has been made in the study of the solution space or real applications of the KP equation. It would appear that the general perception was that there were not many new and significant results left to be uncovered in the soliton solutions of the KP theory. For the last 5 years, we have been working on the classification problem of the soliton solutions of the KP equation [4, 17, 3, 6, 7], and our studies have revealed a large variety of soliton solutions which were totally overlooked in the past. In this paper, we give a brief survey of our research and explain an application of our new soliton solutions to describe the resonant interaction of solitons in shallow water.

The paper is organized as follows:

In Section 2, we describe an interesting connection between the KP equation and the Burgers equation (Theorem 2.1), and then construct some exact solutions based on this connection. We show that the resonant interaction in the KP equation is a natural consequence of the mathematical structure of the KP equation, that is, the Burgers equation gives a symmetry of the KP equation in the y -direction. The Wronskian determinant structure of the solutions is derived from an extension of the Burgers hierarchy (Theorem 2.5, see also [11, 13]).

In Section 3, we present a brief introduction of the Sato theory, and give a direct construction of the Wronskian structure of the τ -function (see also [31, 24]). The KP hierarchy is the set of the symmetries of the KP equation, and the geometric structure of the solutions of the KP hierarchy is given by the τ -function. We explain the important role that the τ -function plays in the integrability of the KP equation.

In Section 4, we give an elementary introduction of the real Grassmann manifold $\text{Gr}(N, M)$, the set of N -dimensional subspaces of \mathbb{R}^M . This is necessary background information for the classification problem of the soliton solutions of the KP equation. In particular, we explain the basic structure of the τ -function in terms of the Schubert decomposition of the Grassmannian $\text{Gr}(N, M)$.

In Section 5, we present the main theorem of the classification problem of the soliton solution (Theorem 5.4). This theorem states that the τ -function identified as a point on $\text{Gr}(N, M)$ generates a solution that has asymptotically $M - N$ line-solitons for $y \ll 0$ and N line-solitons for $y \gg 0$. Moreover, these soliton solutions can be parametrized by the chord diagrams associated with certain permutation called *derangements* of the symmetry group S_M . This type of solutions is called $(M - N, N)$ -soliton solution. We give a new proof of the Theorem, some of which did not appear in the previous papers. Specifically in Corollary 5.1, we show how the conserved densities of the KP equation lead directly to establish the correspondence between the line-soliton solutions and the derangements of S_M .

In Section 6, we study the detailed structure of $(2, 2)$ -solitons defined on $\text{Gr}(2, 4)$ as a fundamental class of soliton solutions of the KP equation. We present the detailed description of the solutions based on the 2×4 matrix which marks the point of $\text{Gr}(2, 4)$. The solution is completely determined if one prescribes a specific data for this matrix. We then show that for some classes of $(2, 2)$ -soliton solutions, the matrix data is equivalent to *asymptotic* data such as the soliton locations and phase shifts, while for some other solution classes, one also need to specify *internal* data given by a local structure.

In Section 7, we discuss an application of our new solutions to the Mach reflection in shallow water with a rigid wall, based on the results of Section 6. The resonant interaction among the incident wave, the reflection wave and the Mach stem is well described by our soliton solution, if we ignore the effect of the boundary layer at the wall. We also present a direct numerical simulation of the KP equation with a V-shape initial wave, which represents the reflection problem of the incident wave on an inclined wall (see also [12, 29, 33]).

2. THE KP EQUATION

The KP equation is the following partial differential equation in $2 + 1$ dimensions,

$$(2.1) \quad (-4u_t + 6uu_x + u_{xxx})_x + 3\beta u_{yy} = 0.$$

When $\beta = 1$, this equation is referred to as the KP II equation, while the equation with $\beta = -1$ is called the KP I equation. Since we consider only the KP II equation in this paper, we simply refer to the KP II as the KP equation throughout the text.

Note when u is independent of y , the well-known Kortweg-de Vries (KdV) equation is obtained from (2.1)

$$-4u_t + 6uu_x + u_{xxx} = 0,$$

which is why the KP equation is regarded as a $(2 + 1)$ -dimensional extension of the KdV equation. Physically, the KP equation was introduced to study the stability of the KdV soliton under the

influence of weak transverse perturbations [15]. So the KP equation should give a better approximation of the original physical system if the system depends weakly on the transverse direction, i.e. the waves are almost parallel to the y -axis.

Let $w = w(x, y, t)$ be a function defined by

$$u = 2w_x,$$

then the KP equation (2.1) takes the form

$$(-4w_{xt} + 12w_x w_{xx} + w_{xxx})_x + 3w_{yy} = 0.$$

Integrating the above equation once with respect to x and setting the integration constant to zero (i.e. w is assumed to be constant as $|x| \rightarrow \infty$), give the *potential* form of the KP equation, namely,

$$(2.2) \quad (-4w_t + 6w_x^2 + w_{xxx})_x + 3w_{yy} = 0.$$

We will henceforth refer (2.2) as the pKP equation for the function w which plays an important role in our description of the soliton solutions.

We next review an interesting connection between the KP equation and the Burgers hierarchy, first established systematically in [13]. This connection is the basis of our description of the fundamental structure of the soliton solutions discussed in this paper.

2.1. The Burgers hierarchy and the KP equation. Let us first briefly review the Cole-Hopf transformation linking the heat (linear) and Burgers hierarchies:

Proposition 2.1. *If f is a solution of the set of linear equations, i.e. the heat hierarchy,*

$$(2.3) \quad \partial_{t_n} f = \partial_x^n f \quad \text{for } n = 1, 2, \dots,$$

then $w := \partial_x(\ln f)$ satisfies the following set of nonlinear equations, i.e. the Burgers hierarchy,

$$(2.4) \quad \partial_{t_n} w = \partial_x(\partial_x + w)^{n-1} w \quad \text{for } n = 1, 2, \dots$$

Proof. First, by induction, we prove that

$$\partial_x^n f = [(\partial_x + w)^{n-1} w] f := P_n f.$$

When $n = 1$, the above formula is just the definition of the function w . Now, suppose $\partial_x^{n-1} f = [(\partial_x + w)^{n-2} w] f := P_{n-1} f$. Then,

$$\partial_x^n f = \partial_x(P_{n-1} f) = \partial_x(P_{n-1}) f + P_{n-1} w f = [(\partial_x + w) P_{n-1}] f = P_n f.$$

Therefore, $\partial_{t_n} w = \partial_{t_n}(\partial_x \ln f) = \partial_x(\partial_{t_n} f f^{-1}) = \partial_x(\partial_x^n f f^{-1}) = \partial_x(\partial_x + w)^{n-1} w$. \square

Note here that the Burgers hierarchy is well-defined in the sense that all the members commute, because the equations for f in (2.3) are commute trivially. The Burgers equation corresponds to $n = 2$ in (2.4), i.e. the first nontrivial member of the hierarchy (2.4), and the members for $n = 2, 3$ are given by

$$(2.5) \quad \begin{cases} \partial_{t_2} w &= \partial_x(w_x + w^2), \\ \partial_{t_3} w &= \partial_x(w_{xx} + 3ww_x + w^3). \end{cases}$$

Note that the even members of the hierarchy are *dissipative*, while the odd ones are *dispersive*. This feature is a key for the resonance phenomena in the soliton solutions of the KP equation (see Example 2.4).

Now we show the connection between the KP equation and the Burgers hierarchy (2.4):

Proposition 2.2. *Suppose $w(x, t_2, t_3)$ is a common solution of (2.5) with $t_2 = y$ and $t_3 = t$, then $w = w(x, y, t)$ solves the pKP equation (2.2).*

Proof. Expressing w_{yy} , w_{xt} in terms of only the x -partial derivatives of w , we have

$$\begin{aligned} w_{yy} &= \partial_x(w_{xxx} + 2w_x^2 + 4ww_{xx} + 4w^2w_x), \\ w_{xt} &= \partial_x(w_{xxx} + 3w_x^2 + 3ww_{xx} + 3w^2w_x). \end{aligned}$$

Then eliminating the common terms including $(ww_{xx} + w^2w_x)$, i.e. calculate $3w_{yy} - 4w_{xt}$, we obtain the pKP equation (2.2). \square

Thus the KP equation contains the Burgers hierarchy, and the next theorem combines the results of Propositions 2.1 and 2.2, which gives a *linearization* of the KP equation:

Theorem 2.1. *If $f(x, t_2, t_3, \dots)$ satisfies the set of linear equations, $\partial_{t_n} f = \partial_x^n f$, $n = 1, 2, \dots$ with $t_2 := y, t_3 := t$, then $w = \partial_x(\ln f)$ satisfies the pKP equation (2.2).*

This implies that a solution of the KP equation obtained from the Burgers hierarchy (equivalently the set of linear equations) has a dissipative behavior in the y -direction. In particular, one should note that a confluence of shocks [34] in the Burgers equation corresponds to a fusion of solitons as shown by Example 2.4 of the next section.

Remark 2.2. One should also note that the infinitely many commuting symmetries of the Burgers hierarchy (2.4) induces higher flows of the pKP hierarchy. In other words, proceeding in the same way as in Proposition 2.2 one could also obtain an equation for $w(x, t_2, \dots)$ in the first n variables x, t_2, t_3, \dots, t_n , which form the n th member of the pKP hierarchy. The substitution $u = 2w_x$ then gives the corresponding member of the KP hierarchy. Thus, the KP equation (2.1) corresponds to the $n = 3$ flow of the KP hierarchy, and the higher flows generate the infinitely many symmetries of the KP equations. The KP hierarchy plays an important role when we discuss the multi-soliton solutions as we show in this paper.

2.2. Some exact solutions. We now consider some solutions of the KP equation obtained from the linear system (2.3). The general solution for this linear system with $t_2 = y$ and $t_3 = t$ admits the integral representation (Ehrenpreis principle)

$$f(x, y, t) = \int_C e^{kx + k^2y + k^3t} \rho(k) dk,$$

with an appropriate measure $\rho(k) dk$ and a proper contour C in the complex plane. A particularly simple finite dimensional solution is recovered by choosing

$$\rho(k) dk = \sum_{j=1}^M a_j \delta(k - k_j) dk$$

with arbitrary real constants k_j and a_j for $j = 1, 2, \dots, M$, and the contour $C = \mathbb{R}$. Then

$$(2.6) \quad f(x, y, t) = \sum_{j=1}^M a_j E_j(x, y, t) \quad E_j(x, y, t) := e^{\theta_j(x, y, t)},$$

with $\theta_j := k_j x + k_j^2 y + k_j^3 t$. Here we assume a_j to be positive, so that f is positive definite, and also assume the ordering in the k -parameters,

$$(2.7) \quad k_1 < k_2 < \dots < k_M.$$

Now let us give some explicit examples:

Example 2.3. Consider the case with $M = 2$, i.e. $f = a_1 E_1 + a_2 E_2$. Since $w = \partial_x(\ln f)$, we can choose $a_1 = 1$ without altering the solution. So we take, with $a_1 = a$,

$$f = E_1 + a E_2 = 2\sqrt{a} e^{\frac{1}{2}(\theta_1 + \theta_2)} \cosh \frac{1}{2}(\theta_1 - \theta_2 + \theta_{12}),$$

where $\theta_{12} = -\ln a$. Then we have

$$u = 2\partial_x^2 \ln f = \frac{1}{2}(k_1 - k_2)^2 \operatorname{sech}^2 \frac{1}{2}(\theta_1 - \theta_2 + \theta_{12}),$$

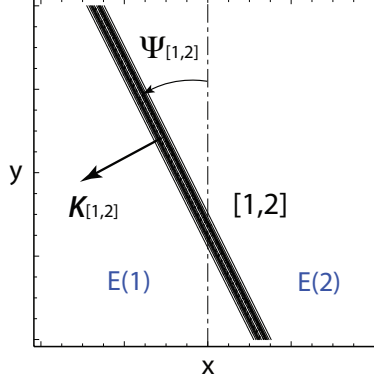


FIGURE 2.1. A One-soliton solution. The $[1, 2]$ is the label of this line-soliton, and each $E(j)$ for $j = 1, 2$ indicates the dominant exponential term E_j in this region. The amplitude $A_{[1,2]}$ and the angle $\Psi_{[1,2]}$ are given by $A_{[1,2]} = \frac{1}{2}(k_2 - k_1)^2$ and $\tan \Psi_{[1,2]} = k_1 + k_2$. $\mathbf{K}_{[1,2]} = (K_{[1,2]}^x, K_{[1,2]}^y)$ shows the wave-vector, and the slope is given by $\tan \Psi_{[1,2]} = K_{[1,2]}^y / K_{[1,2]}^x$. Throughout the paper, the graphs of soliton solutions illustrate contour lines of the function $u(x, y, t)$.

which is called the one-soliton solution. The solution $u(x, y, t)$ is localized in the xy -plane along the line $\theta_1 - \theta_2 + \theta_{12} = 0$, whose location is determined by the constant a . In particular, at $t = 0$ the peak of the soliton determines the line,

$$x + (k_1 + k_2)y = \frac{1}{k_2 - k_1} \ln a,$$

so that choosing $a = 1$ the line of the peak crosses the origin. Because of this, we often refer the one-soliton as a line-soliton (in a local sense in general).

We also note that in terms of the function $w(x, y, t)$, we have the following asymptotic values for $x \rightarrow \pm\infty$ with the ordering $k_1 < k_2$,

$$w = \frac{f_x}{f} = \frac{k_1 E_1 + k_2 E_2}{E_1 + E_2} \rightarrow \begin{cases} k_2 & \text{as } x \rightarrow +\infty, \\ k_1 & \text{as } x \rightarrow -\infty. \end{cases}$$

and the solution $u = 2w_x \rightarrow 0$, exponentially as $|x| \rightarrow \infty$. The change in the asymptotic value of the potential function w for the one-soliton solution is due to fact that the exponential E_1 dominates over E_2 as $x \rightarrow -\infty$, whereas the exponential E_2 dominates over E_1 as $x \rightarrow \infty$. Consequently, the solution u is localized along the phase transition line: $\theta_1 - \theta_2 = \text{constant}$, where both exponential terms are in balance. This motivates labeling the one-soliton solution as the $[1, 2]$ -soliton which also represents a permutation $\pi = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ of the index set $\{1, 2\}$, i.e. exchanging the values k_1 and k_2 by crossing the line-soliton.

It is also convenient to introduce the following notations to describe a *line*-soliton physically as a traveling wave: We say that a line-soliton is of $[i, j]$ -type (or simply $[i, j]$ -soliton), if the soliton solution u has the form (locally),

$$(2.8) \quad u = A_{[i,j]} \operatorname{sech}^2 \left(\mathbf{K}_{[i,j]} \cdot \mathbf{x} + \Omega_{[i,j]} t + \Theta_{[i,j]}^0 \right)$$

The amplitude $A_{[i,j]}$, the wave-vector $\mathbf{K}_{[i,j]}$ and the frequency $\Omega_{[i,j]}$ are defined by

$$\begin{aligned} A_{[i,j]} &= \frac{1}{2}(k_j - k_i)^2 \\ \mathbf{K}_{[i,j]} &= \left(\frac{1}{2}(k_j - k_i), \frac{1}{2}(k_j^2 - k_i^2) \right) = \frac{1}{2}(k_j - k_i)(1, k_i + k_j), \\ \Omega_{[i,j]} &= \frac{1}{2}(k_j^3 - k_i^3) = \frac{1}{2}(k_j - k_i)(k_i^2 + k_i k_j + k_j^2). \end{aligned}$$

The direction of the wave-vector $\mathbf{K}_{[i,j]} = (K_{[i,j]}^x, K_{[i,j]}^y)$ is measured in the counterclockwise from the x -axis (see Figure 2.1), and it is given by

$$\frac{K_{[i,j]}^y}{K_{[i,j]}^x} = \tan \Psi_{[i,j]} = k_i + k_j.$$

Note here that the $\Psi_{[i,j]}$ also gives the angle of the line $[i, j]$ from the y -axis in the counterclockwise, and $-\frac{\pi}{2} < \Psi_{[i,j]} < \frac{\pi}{2}$. For each soliton solution of (2.8), the wave vector $\mathbf{K}_{[i,j]}$ and the frequency $\Omega_{[i,j]}$ satisfy the soliton-dispersion relation,

$$(2.9) \quad 4\Omega_{[i,j]}K_{[i,j]}^x = 4(K_{[i,j]}^x)^4 + 3(K_{[i,j]}^y)^2.$$

The soliton velocity $\mathbf{V}_{[i,j]}$ defined by $\mathbf{K}_{[i,j]} \cdot \mathbf{V}_{[i,j]} = -\Omega_{[i,j]}$ is given by

$$\mathbf{V}_{[i,j]} = -\frac{\Omega_{[i,j]}}{|\mathbf{K}_{[i,j]}|^2} \mathbf{K}_{[i,j]} = \frac{k_i^2 + k_i k_j + k_j^2}{1 + (k_i + k_j)^2} (-1, -(k_i + k_j)).$$

Note in particular that $(k_i^2 + k_i k_j + k_j^2) > 0$, and this implies that the x -component of the velocity is *always* negative. That is, a line-soliton propagates in the negative x -direction. On the other hand, any small perturbation propagates in the positive x -direction. This can be seen from the dispersion relation of the KP equation for a linear wave $\phi = \exp(i\mathbf{k} \cdot \mathbf{x} - i\omega t)$ with wave-vector $\mathbf{k} = (k_x, k_y)$ and frequency ω ,

$$\omega = \frac{1}{4}k_x^3 - \frac{3}{4}\frac{k_y^2}{k_x},$$

from which the group velocity of the wave is given by

$$\mathbf{v} = \nabla \omega = \left(\frac{\partial \omega}{\partial k_x}, \frac{\partial \omega}{\partial k_y} \right) = \left(\frac{3}{4} \left(k_x^2 + \frac{k_y^2}{k_x^2} \right), -\frac{2}{3} \frac{k_y}{k_x} \right).$$

Thus, it is reasonable to expect from a first-order perturbation theory that the soliton separates from small radiations asymptotically, similar to the case of the KdV equation (see for example [1, 25]).

We also remark that the formula (2.8) for one-soliton solution of the KP equation can be extended to the one-soliton solution of the KP hierarchy by including the higher times t_n (see Remark 2.2),

$$(2.10) \quad u(t_1, t_2, t_3, \dots) = \mathcal{A} \operatorname{sech}^2 \left(\sum_{n=1}^{\infty} \mathcal{K}_n t_n + \Theta^0 \right),$$

with some constant Θ^0 . Here the amplitude and the *infinite* dimensional wave-vector are given by

$$\begin{cases} \mathcal{A} &= \frac{1}{2}(k_j - k_i)^2, \\ \mathcal{K}_n &= \frac{1}{2}(k_j^n - k_i^n), \quad \text{for } n = 1, 2, \dots \end{cases}$$

This can be easily seen from the structure of f -function, i.e.

$$f = E_i + aE_j, \quad \text{with } E_i = \exp \left(\sum_{n=1}^{\infty} k_i^n t_n \right).$$

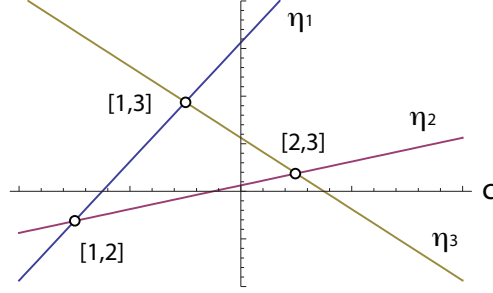


FIGURE 2.2. The graphs of $\eta_j(c) = k_j(k_j - c)$. Each $[i, j]$ represents the exchange of the order between η_i and η_j . The parameters are given by $(k_1, k_2, k_3) = (-\frac{5}{4}, -\frac{1}{4}, \frac{3}{4})$.

Although the higher times t_n for $n > 3$ do not have a direct physical meaning, those parameters are related to the existence of the multi-soliton solutions through the symmetries of the KP equation as mentioned in Remark 2.2.

Example 2.4. Now we consider the case with $M = 3$. We again take $a_1 = 1$, so that we have

$$f = E_1 + aE_2 + bE_3$$

with some positive constants a and b . As in the previous example, it is also possible here to determine the dominant exponentials and analyze the structure of the solution in the xy -plane. Let us consider the function f along the line $x = -cy$ with $c = \tan \Psi$ where Ψ is the angle measured counterclockwise from the y -axis (see Figure 2.1). Then along $x = -cy$, we have the exponential function $E_j = \exp[\eta_j(c)y + k_j^3 t]$ with

$$(2.11) \quad \eta_j(c) = k_j(k_j - c).$$

It is then seen from Figure 2.2 that for $y \gg 0$ and a fixed t , the exponential term E_1 dominates when c is large positive ($\Psi \approx \frac{\pi}{2}$, i.e. $x \rightarrow -\infty$). Decreasing the value of c (rotating the line clockwise), the dominant term changes to E_3 . Thus we have

$$w = \partial_x \ln f \longrightarrow \begin{cases} k_1 & \text{as } x \rightarrow -\infty, \\ k_3 & \text{as } x \rightarrow \infty. \end{cases}$$

The transition of the dominant exponentials $E_1 \rightarrow E_3$ is characterized by the condition $\eta_1 = \eta_3$, which corresponds the direction parameter value $c = \tan \Psi_{[1,3]} = k_1 + k_3$. In the neighborhood of this line, the function f can be approximated as

$$f \approx E_1 + bE_3,$$

which implies that there exists a $[1, 3]$ -soliton for $y \gg 0$. The constant b can be used to choose a specific location of this soliton.

Next consider the case of $y \ll 0$. The dominant exponential corresponds to the *least* value of η_j for any given value of c . For large positive c ($\Psi \approx \frac{\pi}{2}$, i.e. $x \rightarrow \infty$), E_3 is the dominant term. Decreasing the value of c (rotating the line $x = -cy$ clockwise), the dominant term changes to E_2 when $k_2 + k_3 > c > k_1 + k_2$, and E_1 becomes dominant for $c < k_1 + k_2$. Hence, we have for $y \ll 0$

$$w \longrightarrow \begin{cases} k_1 & \text{as } x \rightarrow -\infty, \\ k_2 & \text{for } -(k_1 + k_2)y < x < -(k_2 + k_3)y, \\ k_3 & \text{as } x \rightarrow \infty. \end{cases}$$

In the neighborhood of the line $x + (k_1 + k_2)y = \text{constant}$,

$$f \approx E_1 + aE_2,$$

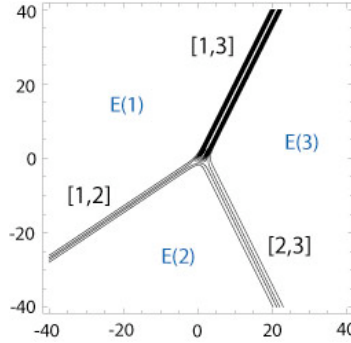


FIGURE 2.3. A $(2,1)$ -soliton solution. Each $E(j)$ with $j = 1, 2$ or 3 indicates the dominant exponential term E_j in that region. The boundaries of any two adjacent regions give the line-solitons indicating the transition of the dominant terms E_j . The k -parameters are the same as those in Figure 2.2, and the line-solitons are determined from the intersection points of the $\eta_j(c)$'s in Figure 2.2. Here $a = b = 1$ (i.e. $\tau = E_1 + E_2 + E_3$) so that the three solitons meet at the origin at $t = 0$.

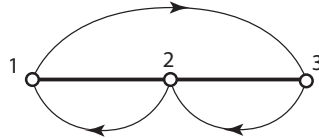
which corresponds to a $[1,2]$ -soliton and its location is fixed by the constant a . The solutions also consists of a $[2,3]$ -soliton in the neighborhood of the line $x + (k_2 + k_3)y = \text{constant}$, and whose location is determined by the locations of other line-solitons. Therefore, we need only two parameters a, b (besides the k -parameters) to specify the solution uniquely. These free parameters may be considered as the asymptotic data for the solution u . The shape of solution generated by $f = E_1 + aE_2 + bE_3$ with $a = b = 1$ (i.e. at $t = 0$ three line-solitons meet at the origin) is illustrated via the contour plot in Figure 2.3. In this Figure, one can see that the line-soliton in $y \gg 0$ labeled by $[1,3]$, is localized along the phase transition line $\theta_1 = \theta_3$ (equivalently, $\eta_1 = \eta_3$) with direction parameter $c = k_1 + k_3$; two other line-solitons in $y \ll 0$ labeled by $[1,2]$ and $[2,3]$ are localized respectively, along the phase transition lines with $c = k_1 + k_2$ and $c = k_2 + k_3$. This solution represents a resonant solution of three line-solitons. In terms of the function w , which is a solution of the Burgers equation in the y -direction, this corresponds to a confluence of two shocks (see p.110 in [34]). The resonant condition among those three line-solitons is given by

$$\mathbf{K}_{[1,3]} = \mathbf{K}_{[1,2]} + \mathbf{K}_{[2,3]}, \quad \Omega_{[1,3]} = \Omega_{[1,2]} + \Omega_{[2,3]},$$

which are identically satisfied with $\mathbf{K}_{[i,j]} = \frac{1}{2}(k_j - k_i, k_j^2 - k_i^2)$ and $\Omega_{[i,j]} = \frac{1}{2}(k_j^3 - k_i^3)$. The resonant condition may be symbolically written as

$$[1,3] = [1,2] + [2,3].$$

One can also represent this line-soliton solution by a permutation of three indices: $\{1, 2, 3\}$ which is illustrated by a (linear) *chord diagram* shown below. Here, the upper chord represents the $[1,3]$ -



soliton in $y \gg 0$ and the lower two chords represent $[1,2]$ and $[2,3]$ -solitons in $y \ll 0$. Following the arrows in the chord diagram, one recovers the permutation,

$$\pi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad \text{or simply} \quad \pi = (312).$$

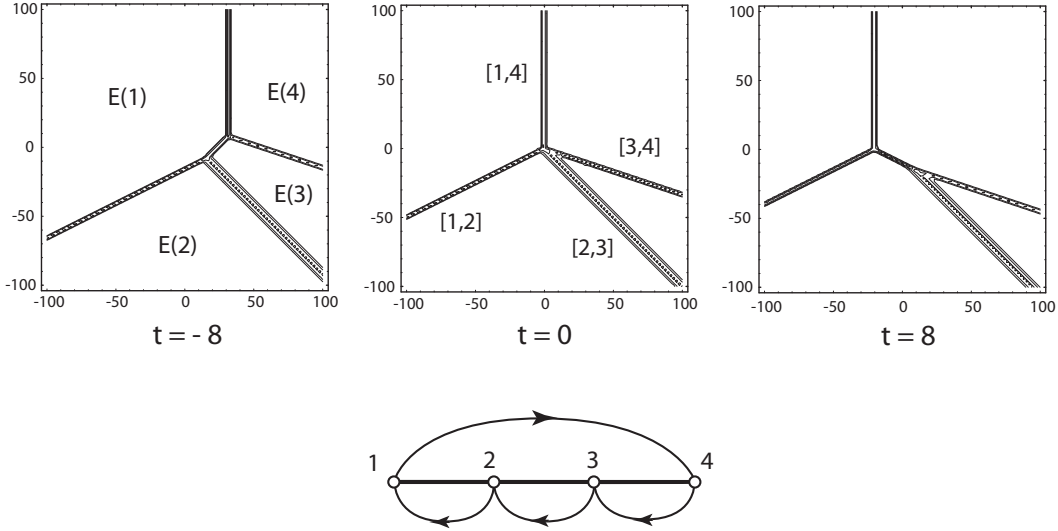


FIGURE 2.4. The time evolution of a $(3,1)$ -soliton solution and the corresponding chord diagram. The upper chord represents the $[1,4]$ -soliton, and the lower ones represent $[1,2]$ -, $[2,3]$ - and $[3,4]$ -solitons. The chord diagram shows $\pi = (4123)$.

In general, each line-soliton solution of the KP equation can be parametrized by a *unique* permutation corresponding to a chord diagram. We will discuss this issue in Section 5.

The results described in the previous examples can be easily extended to the general case where f has arbitrary number of exponential terms (see also [21, 4]).

Proposition 2.3. *If $f = a_1 E_1 + a_2 E_2 + \cdots + a_M E_M$ with $a_j > 0$ for $j = 1, 2, \dots, M$, then the solution u consists of $M - 1$ line-solitons for $y \ll 0$ and one line-soliton for $y \gg 0$.*

Such solutions are referred to as the $(M - 1, 1)$ -soliton solutions. Note that the line-soliton for $y \gg 0$ is labeled by $[1, M]$, whereas the other line-solitons in $y \ll 0$ are labeled by $[k, k + 1]$ for $k = 1, 2, \dots, M - 1$, counterclockwise from the negative to the positive x -axis, i.e. increasing Ψ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. As in the previous examples one can set $a_1 = 1$ without any loss of generality, then the remaining $M - 1$ parameters a_2, \dots, a_M determine the locations of the M line-solitons. Also note that the xy -plane is divided into M sectors for the asymptotic region with $x^2 + y^2 \gg 0$, and the boundaries of those sectors are given by the asymptotic line-solitons. This feature is common even for the general case. Figure 2.4 illustrates the case for a $(3, 1)$ -soliton solution with $f = E_1 + E_2 + E_3 + E_4$. The chord diagram for this solution represents the permutation $\pi = (4123)$ of the set $\{1, 2, 3, 4\}$.

2.3. Extension of the Burgers hierarchy. Theorem 2.1 shows the relation between the KP equation and the Burgers hierarchy for the function w defined in terms of the solution f of the linear system (2.3) via the Cole-Hopf transformation,

$$(2.12) \quad w = \partial_x \ln f, \quad \text{or} \quad f^{(1)} = w f,$$

where $f^{(1)} = \partial_x f$. Furthermore, Proposition 2.3 illustrates the $(M - 1, 1)$ -soliton solutions obtained from the special choices for the functions f and w . In order to construct more general type of soliton solutions of the KP equation, we consider an extension of the Cole-Hopf transformation (2.12) to the case where the function f satisfies an N -th order linear equation,

$$(2.13) \quad f^{(N)} = w_1 f^{(N-1)} + \cdots + w_{N-1} f^{(1)} + w_N f,$$

where $f^{(j)} := \partial_x^j f$. The coefficient functions (w_1, w_2, \dots, w_N) can be constructed from N linearly independent solutions of (2.13), and their time evolutions with respect to t_n , $n > 1$ are determined

by the compatibility conditions of (2.13) with the linear system (2.3). For example, when $N = 2$, the compatibility condition $\partial_y(f_{xx}) = \partial_x^2(f_y)$ with $y = t_2$ gives $\partial_y(w_1 f^{(1)} + w_2 f) = \partial_x^2(w_1 f^{(1)} + w_2 f)$. Then by equating the coefficients of $f^{(1)}$ and f in the last expression, leads to the following equations for (w_1, w_2) ,

$$\begin{aligned}\partial_y w_1 &= 2w_1 w_{1,x} + w_{1,xx} + 2w_{2,x}, \\ \partial_y w_2 &= 2w_2 w_{1,x} + w_{2,xx}.\end{aligned}$$

Notice that if $w_2 = 0$, this system is reduced to the Burgers equation. In general, the variables (w_1, w_2, \dots, w_N) satisfy an N -component Burgers equation in the y -variable while the higher flows with respect to the “times” t_n for $n = 3, 4, \dots$ are the symmetries of this coupled equation, thus forming an N -component Burgers hierarchy (this will be reformulated in terms of the Sato theory in the next section, also see [13]).

We regard (2.13) as an N -th order *ordinary* differential equation, and consider a fundamental set of N solutions denoted by $\{f_1, f_2, \dots, f_N\}$, which satisfy

$$f_i^{(N)} = w_1 f_i^{(N-1)} + \dots + w_N f_i, \quad i = 1, 2, \dots, N.$$

The above form a linear algebraic system which can be solved for the coefficients $\{w_1, w_2, \dots, w_N\}$ by applying Cramer’s formula. In particular we have

$$\begin{aligned}w_1 &= \frac{1}{\text{Wr}(f_1, \dots, f_N)} \begin{vmatrix} f_1 & f_1^{(1)} & \dots & f_1^{(N-2)} & f_1^{(N)} \\ f_2 & f_2^{(1)} & \dots & f_2^{(N-2)} & f_2^{(N)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_N & f_N^{(1)} & \dots & f_N^{(N-2)} & f_N^{(N)} \end{vmatrix} \\ &= \frac{\partial}{\partial x} \ln \text{Wr}(f_1, \dots, f_N).\end{aligned}$$

where $\text{Wr}(f_1, \dots, f_N)$ is the Wronskian determinant of $\{f_1, \dots, f_N\}$. For $N = 1$, this is just the Cole-Hopf transformation (2.12) where $w_1 = w$ solves the pKP equation. In what follows, we show that this is also true for the general case. That is, the function $w_1 = \partial_x \ln \text{Wr}(f_1, \dots, f_N)$ solves the pKP equation, and $u := 2w_{1,x} = 2\partial_x^2 \ln \text{Wr}(f_1, \dots, f_N)$ solves the KP equation. The $\text{Wr}(f_1, \dots, f_N)$ is an example of the τ -function for the KP equation, which plays a very important role of the KP theory (see Section 3.3). We now state the well-known theorem that leads to the Wronskian formulation of the multi-soliton solution for the KP equation.

Theorem 2.5. *Let the τ -function be given by the Wronskian determinant,*

$$(2.14) \quad \tau(x, y, t) = \begin{vmatrix} f_1 & f_1^{(1)} & \dots & f_1^{(N-1)} \\ f_2 & f_2^{(1)} & \dots & f_2^{(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ f_N & f_N^{(1)} & \dots & f_N^{(N-1)} \end{vmatrix},$$

where $\{f_1, f_2, \dots, f_N\}$ are linearly independent functions satisfying $\partial_y f_j = f_j^{(2)}$ and $\partial_t f_j = f_j^{(3)}$ with $f_j^{(n)} := \partial_x^n f_j$ for $j = 1, \dots, N$. Then $u(x, y, t) = 2\partial_x^2 \ln \tau(x, y, t)$ satisfies the KP equation (2.1).

We prove the theorem by employing an identity called the Plücker relation (see for example [14]), satisfied by the maximal minors of an $N \times M$ matrix

$$\Phi := \begin{pmatrix} f_1 & f_1^{(1)} & \dots & f_1^{(M-1)} \\ f_2 & f_2^{(1)} & \dots & f_2^{(M-1)} \\ \vdots & \vdots & \ddots & \vdots \\ f_N & f_N^{(1)} & \dots & f_N^{(M-1)} \end{pmatrix}.$$

with $N < M$. A maximal minor $\varphi(l_1, l_2, \dots, l_N)$ is the determinant of the submatrix formed by N columns $\Phi_i = (f_1^{(i-1)}, f_2^{(i-1)}, \dots, f_N^{(i-1)})^T$ of Φ where $i \in \{l_1, l_2, \dots, l_N\}$, i.e.

$$\varphi(l_1, l_2, \dots, l_N) = \det [\Phi_{l_1}, \Phi_{l_2}, \dots, \Phi_{l_N}].$$

Lemma 2.1. *The set of maximal minors $\varphi[i, j] := \varphi(1, 2, \dots, N-2, N-2+i, N-2+j)$ of the matrix Φ defined above, with $1 \leq i < j \leq 4$, satisfy the identity,*

$$(2.15) \quad \varphi[1, 2] \varphi[3, 4] - \varphi[1, 3] \varphi[2, 4] + \varphi[1, 4] \varphi[2, 3] = 0.$$

Proof. Consider the following $2N \times 2N$ determinant expressed in terms of the columns Φ_i of the matrix Φ ,

$$\det \begin{bmatrix} \Phi_1 & \cdots & \Phi_{N-1} & \Phi_1 & \cdots & \Phi_{N-2} & \Phi_N & \Phi_{N+1} & \Phi_{N+2} \\ 0 & \cdots & 0 & \Phi_1 & \cdots & \Phi_{N-2} & \Phi_N & \Phi_{N+1} & \Phi_{N+2} \end{bmatrix}.$$

This determinant is obviously zero. Then, Laplace expansion of the first determinant in terms of $N \times N$ minors yields the desired identity. \square

The three-term identity (2.15) is one of the Plücker relations which play an important role in revealing the geometric structure underlying the KP equation. This will be discussed in Section 4, where we will introduce the general form of the Plücker relations. We now prove Theorem 2.5 (the following proof is the same as that in [14]):

Proof. Substituting the relation $u = 2\partial_x^2(\ln \tau)$ into the KP equation (2.1), integrating twice with respect to x and setting the integrations constants to zero, yields the following equation in τ ,

$$(2.16) \quad 4(\tau\tau_{xt} - \tau_x\tau_t) - 3\tau_{xx}^2 - \tau\tau_{xxxx} + 4\tau_x\tau_{xxx} + 3\tau_y^2 - 3\tau\tau_{yy} = 0.$$

The τ -function is given by the maximal minor, $\tau = \varphi(1, 2, \dots, N) =: \varphi[1, 2]$, using the notation of Lemma 2.1. Suitable combinations of the derivatives of τ also correspond to other maximal minors of the matrix Φ . In particular, we have

$$\begin{aligned} \varphi[1, 3] &= \tau_x, & \varphi[1, 4] &= \frac{1}{2}(\tau_{xx} + \tau_y), & \varphi[2, 3] &= \frac{1}{2}(\tau_{xx} - \tau_y), \\ \varphi[2, 4] &= \frac{1}{3}(\tau_{xxx} - \tau_t), & \varphi[3, 4] &= \frac{1}{12}(\tau_{xxxx} + 3\tau_{yy} - 4\tau_{xt}) \end{aligned}$$

Substituting the above expressions into the Plücker relation (2.15) in Lemma 2.1, we have

$$\begin{aligned} 0 &= \varphi[1, 2] \varphi[3, 4] - \varphi[1, 3] \varphi[2, 4] + \varphi[1, 4] \varphi[2, 3] \\ &= -\frac{1}{12}(\tau(4\tau_{xt} - \tau_{xxxx} - 3\tau_{yy}) + 4\tau_x(\tau_{xxx} - \tau_t) + 3(\tau_y + \tau_{xx})(\tau_y - \tau_{xx})) \end{aligned}$$

which is equivalent to (2.16). \square

Theorem 2.5 provides a direct *linearization* scheme for the KP equation in the sense that a large class of solutions whose τ -function is given by the Wronskian determinant (2.14) can be constructed from the solutions of the linear equation (2.13) together with the evolution equations $f_y = f_{xx}$ and $f_t = f_{xxx}$. Note that since $f(x, y, t)$ admits a rather general integral representation (see Section 2.2), this linearization scheme gives rise to different classes of solution besides the soliton solutions, by proper choice of measure and contour of integration in the k -plane. It is a remarkable fact that all solutions constructed in this way identically satisfy the KP equation by virtue of the Plücker relation (2.15) which is an algebraic identity. However, it should be emphasized that the significance of Theorem 2.5 is much deeper than that is presented here. In the original scope of Sato theory, the differential operator in (2.13) is in fact a reduction (N -truncation) of a pseudo-differential operator of infinite order (see Section 3) all of whose coefficients are determined via a single holomorphic function namely, the Sato τ -function. The Wronskian determinant (2.14) is a representation of the Sato τ -function in the N -truncated case. It is in this (infinite-dimensional) setting of the full Sato theory that the KP equation expressed in terms of the Sato τ -function as in (2.16) can itself be interpreted as the Plücker relation (2.15) in a suitable sense (see also Remark 4.4).

Remark 2.6. The equation (2.16) is often rewritten in the Hirota bi-linear form

$$P(D_x, D_y, D_t)\tau \cdot \tau := (4D_x D_t - D_x^4 - 3D_y^2)\tau \cdot \tau = 0,$$

where D_z with $z = x, y$ or t is the usual Hirota derivative defined by

$$D_z^n f \cdot g := \frac{\partial^n}{\partial \epsilon^n} [f(z + \epsilon)g(z - \epsilon)] \Big|_{\epsilon=0}.$$

Note here that the soliton dispersion relation (2.9) can be obtained from the Hirota bi-linear form, i.e. $P(2K_{[i,j]}^x, 2K_{[i,j]}^y, 2\Omega_{[i,j]}) = 0$. Thus the dispersion relation has a direct connection to the Plücker relation (2.15).

In his famous book [14], Hirota expresses the minors $\phi[i, j]$ with the elegant diagrams called the Maya diagrams introduced by Sato (which is also equivalent to the Young diagrams). Explicit proofs of the solution formulas for other integrable systems written in the Hirota bi-linear forms are also given in terms of the Maya diagrams.

Example 2.7. Consider the case with $N = 2$ and $M = 3$: The linearly independent functions f_1 and f_2 are expressed as,

$$f_i = \sum_{j=1}^3 a_{ij} E_j, \quad i = 1, 2,$$

with $E_j(x, y, t) = \exp(k_j x + k_j^2 y + k_j^3 t)$ for $j = 1, 2, 3$. In this case the τ -function in (2.14) can be explicitly given by

$$\begin{aligned} \tau &= \begin{vmatrix} f_1 & f_1^{(1)} \\ f_2 & f_2^{(1)} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{vmatrix} \begin{vmatrix} E_1 & k_1 E_1 \\ E_2 & k_2 E_2 \\ E_3 & k_3 E_3 \end{vmatrix} \\ &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} E(1, 2) + \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} E(1, 3) + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} E(2, 3), \end{aligned}$$

where we have used the Binet-Cauchy theorem and $E(i, j) = \text{Wr}(E_i, E_j) = (k_j - k_i)E_i E_j$. Let us investigate a concrete situation where $(k_1, k_2, k_3) = (-\frac{5}{4}, -\frac{1}{4}, \frac{3}{4})$, and

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -b \\ 0 & 1 & a \end{pmatrix},$$

with positive constants a and b . The τ -function is then given by

$$\tau = E(1, 2) + aE(1, 3) + bE(2, 3).$$

In order to carry out the asymptotic analysis in this case one needs to consider the sum of two $\eta_j(c)$, i.e. $\eta_{i,j} = \eta_i + \eta_j$ for $1 \leq i < j \leq 3$. This can still be done using Figure 2.2, but a more effective way is described in Section 5 (see the graph of $\eta(k, c)$ in Figure 5.1 and equations (5.4), (5.5)).

For $y \gg 0$, the transitions of the dominant exponentials are given by following scheme:

$$E(1, 2) \longrightarrow E(1, 3) \longrightarrow E(2, 3),$$

as c varies from large positive (i.e. $x \rightarrow -\infty$) to large negative values (i.e. $x \rightarrow \infty$). The boundary between the regions with the dominant exponentials $E(1, 2)$ and $E(1, 3)$ defines the $[2, 3]$ -soliton solution since here the τ -function can be approximated as

$$\begin{aligned} \tau &\approx E(1, 2) + aE(1, 3) = (k_2 - k_1)E_1 \left(E_2 + a \frac{k_3 - k_1}{k_2 - k_1} E_3 \right) \\ &= 2(k_2 - k_1)E_1 e^{\frac{1}{2}(\theta_2 + \theta_3 - \theta_{23})} \cosh \frac{1}{2}(\theta_2 - \theta_3 + \theta_{23}), \end{aligned}$$

so that we have

$$u = 2\partial_x^2 \ln \tau \approx \frac{1}{2}(k_2 - k_3)^2 \text{sech}^2 \frac{1}{2}(\theta_2 - \theta_3 + \theta_{23}),$$

where θ_{23} is given by

$$\theta_{23} = \ln \frac{k_2 - k_1}{k_3 - k_1} - \ln a \quad \text{i.e.} \quad a = \frac{k_2 - k_1}{k_3 - k_1} e^{-\theta_{23}}.$$

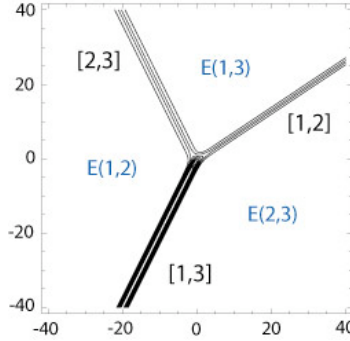


FIGURE 2.5. A $(1,2)$ -soliton solution. The $E(i,j)$ on each region indicates the dominant exponential term. The k -parameters are the same as those in Figure 2.2. The parameters in the A -matrix are chosen as $a = \frac{1}{2}$ and $b = 1$, so that three line-solitons meet at the origin at $t = 0$.

Note here that since the logarithm of the coefficient of cosh-function in τ above is linear in x , it does not contribute to the solution u . The constant a can be used to determine the location of this soliton. A similar computation as above near the transition boundary of the dominant exponentials $E(1,3)$ and $E(2,3)$ yields

$$\begin{aligned}\tau &\approx 2(k_3 - k_1)aE_3e^{\frac{1}{2}(\theta_1 + \theta_2 - \theta_{12})} \cosh \frac{1}{2}(\theta_1 - \theta_2 + \theta_{12}), \\ u &\approx \frac{1}{2}(k_1 - k_2)^2 \operatorname{sech}^2 \frac{1}{2}(\theta_1 - \theta_2 + \theta_{12}),\end{aligned}$$

where θ_{12} is given by

$$\theta_{12} = \ln \frac{k_3 - k_1}{k_3 - k_2} - \ln \frac{b}{a} \quad \text{i.e.} \quad b = \frac{k_2 - k_1}{k_3 - k_2} e^{-\theta_{12}}.$$

This gives $[1,2]$ -soliton and its location is determined by the constant b .

For $y \ll 0$, there is only one transition, namely

$$E(2,3) \longrightarrow E(1,2),$$

as c varies from large positive value (i.e. $x \rightarrow \infty$) to large negative value (i.e. $x \rightarrow -\infty$). In this case, a $[1,3]$ -soliton is formed for $y \ll 0$ at the boundary of the dominant exponentials $E(2,3)$ and $E(1,2)$. The contour plot of the line-soliton solution is shown in Figure 2.5. Notice that this figure can be obtained from Figure 2.3 by changing $(x, y) \rightarrow (-x, -y)$. This solution can be represented by the chord diagram corresponding to the permutation $\pi = (231)$ shown below. Note that this

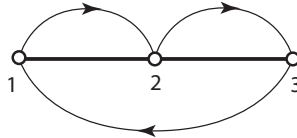


diagram is the π -rotation of the chord diagram in Example 2.4 whose permutation $\pi = (312)$ is the inverse of $\pi = (231)$.

3. SATO THEORY OF THE KP HIERARCHY

In this section we review briefly some basic facts from the Sato theory of the KP hierarchy (see also [8, 24]). We first review the Lax formalism for an infinite set of evolution equations whose compatibility conditions give rise to the flows corresponding to the KP hierarchy. The KP equation

is the first nontrivial member of this hierarchy associated with the time variables $t_1 := x$, $t_2 := y$ and $t_3 := t$. The main purpose is to highlight the basic framework of integrability underlying the KP theory. Then we show that the results of the previous section are direct consequences of the Sato theory. In particular, we emphasize the importance of the τ -function and explain the central role of the τ -function in the KP hierarchy.

3.1. Lax formulation of the KP equation. Let L be a first order pseudo-differential symbol,

$$L = \partial + u_1 + u_2 \partial^{-1} + u_3 \partial^{-2} + \dots$$

where the coefficients $u_i = u_i(\mathbf{t})$ depend on infinitely many variables $\mathbf{t} = (t_1, t_2, t_3, \dots)$. The symbol $\partial := \partial_x$ is a differential operator whereas ∂^{-1} is a formal integration, satisfying $\partial \partial^{-1} = \partial^{-1} \partial = 1$. The operation of ∂^ν with $\nu \in \mathbb{Z}$ is given by the generalized Leibnitz rule

$$\partial^\nu f = \sum_{j \geq 0} \binom{\nu}{j} \partial_x^j(f) \partial^{\nu-j}.$$

For example, we have $\partial f = f_x + f \partial$ and $\partial^{-1} f = f \partial^{-1} - f_x \partial^{-2} + f_{xx} \partial^{-3} - \dots$ (the latter expression follows from the usual integration by parts formula). Here we define the *weights* for the functions u_i and ∂^ν as

$$\text{wt}(u_i) = i, \quad \text{wt}(\partial^\nu) = \nu.$$

So the L has a homogeneous weight one. We also remark that the term u_1 in L can be eliminated by a gauge transformation g such that $u_1 = -g^{-1} \partial_x(g)$, i.e.

$$L \longrightarrow g^{-1} L g = \partial + \tilde{u}_2 \partial^{-1} + \tilde{u}_3 \partial^{-2} + \dots$$

(This can be extended so that we can eliminate *all* u_j with an appropriate gauge operator $g = W$, which will be discussed in Section 3.2.) We will henceforth consider L without the u_1 term,

$$(3.1) \quad L = \partial + u_2 \partial^{-1} + u_3 \partial^{-2} + \dots$$

The Lax form of the KP hierarchy is defined by the infinite set of nonlinear equations

$$(3.2) \quad \partial_{t_n} L = [B_n, L] \quad \text{with} \quad B_n = (L^n)_{\geq 0} \quad n = 1, 2, \dots,$$

where $(L^n)_{\geq 0}$ represents the polynomial part of L^n in ∂ , i.e. B_n is a differential operator of order n , and $[B_n, L] := B_n L - L B_n$ is the usual commutator of operators. Since $[B_n, L] = [L^n - (L^n)_{<0}, L] = [L, (L^n)_{<0}]$, each side of the equation (3.2) is a pseudo-differential operators of order ≤ -1 . Here $(L^n)_{<0}$ is the negative part of L^n in ∂ , and note $[\partial, (L^n)_{<0}]$ has no polynomial part. Thus for $n > 1$, each equation in (3.2) is a consistent but infinite system of coupled $(1+1)$ -evolution equations in t_n and x , for the variables $\{u_i : i = 2, 3, \dots\}$. The case $n = 1$ yields the equations $\partial_{t_1} u_i = \partial_x u_i$, so we identify t_1 with x . The infinite system (3.2) is compatible, as prescribed by the following theorem:

Theorem 3.1. *The differential operators $B_n = (L^n)_{\geq 0}$ satisfy*

$$(3.3) \quad \partial_{t_m} B_n - \partial_{t_n} B_m + [B_n, B_m] = 0.$$

Consequently, the flows defined by (3.2) commute i.e., for any $n, m \geq 1$,

$$\partial_{t_n}(\partial_{t_m} L) = \partial_{t_m}(\partial_{t_n} L).$$

Proof. It follows from (3.2) that $\partial_{t_m}(L^n) = [B_m, L^n]$. Then using the decomposition $L^n = B_n + (L^n)_{<0}$, we have

$$\begin{aligned} \partial_{t_m}(L^n) - \partial_{t_n}(L^m) &= [B_m, L^n] - [B_n, L^m] \\ &= [B_m, B_n] - [(L^m)_{<0}, (L^n)_{<0}]. \end{aligned}$$

The differential part (≥ 0) of the above equation gives (3.3).

To prove that the flows commute we compute using (3.2) once again, and obtain

$$\begin{aligned} \partial_{t_m}(\partial_{t_n} L) - \partial_{t_n}(\partial_{t_m} L) &= [\partial_{t_m} B_n, L] + [B_n, \partial_{t_m} L] - [\partial_{t_n} B_m, L] - [B_m, \partial_{t_n} L] \\ &= [\partial_{t_m} B_n - \partial_{t_n} B_m, L] + [B_n, [B_m, L]] - [B_m, [B_n, L]]. \end{aligned}$$

Applying the Jacobi identity for commutators, the right hand side of the above equations becomes $[\partial_{t_m} B_n - \partial_{t_n} B_m + [B_n, B_m], L]$, which vanishes due to (3.3). \square

Equations (3.3) are called the Zakharov-Shabat equations for the KP hierarchy. Note that for a given pair (n, m) with $n > m$, (3.3) gives a closed system of $n - 1$ equations for u_2, u_3, \dots, u_n , in the variables t_m, t_n and x . For example, consider the case with $n = 3$ and $m = 2$, i.e. $B_2 = (L^2)_{\geq 0} = \partial^2 + 2u_2$ and $B_3 = (L^3)_{\geq 0} = \partial^3 + 3u_2 + 3(u_{2,x} + u_3)$, then the Zakharov-Shabat equation (3.3) for B_2 and B_3 gives the system

$$\begin{cases} \partial_{t_2} u_2 = u_{2,xx} + 2u_{3,x} \\ 2\partial_{t_3} u_2 = 3(u_{2,x} + u_3)_{t_2} - (u_{2,xx} - 3u_{3,x} + 3u_2^2)_x \end{cases}$$

After setting $t_2 = y, t_3 = t$, and eliminating u_3 from the system, $u = 2u_2$ satisfies the KP equation (2.1).

We also remark that the KP hierarchy (3.2) is given by the compatibility of the linear system

$$(3.4) \quad \begin{cases} L\phi &= k\phi, \\ \partial_{t_n}\phi &= B_n\phi, \end{cases} \quad n = 1, 2, \dots,$$

where $k \in \mathbb{C}$, the spectral parameter, and $\phi(\mathbf{t}; k)$ is the eigenfunction of the KP hierarchy. The compatibility among the second set of equations gives the Zakharov-Shabat equations.

3.2. The dressing transformation. As we mentioned in the previous section, the Lax operator L can be gauge-transformed into the trivial operator ∂ , i.e.

$$(3.5) \quad L \longrightarrow \partial = W^{-1}LW,$$

where the operator of the gauge transformation is defined by

$$(3.6) \quad W = 1 - w_1\partial^{-1} - w_2\partial^{-2} - w_3\partial^{-3} + \dots.$$

The coefficient functions w_i are related to the coefficients u_j of L via the relation $LW = W\partial$, and which yields

$$\begin{aligned} u_2 &= w_{1,x}, \quad u_3 = w_{2,x} + w_1w_{1,x}, \quad u_4 = w_{3,x} + (w_1w_2)_x - w_{1,x}^2 + w_1^2w_{1,x}, \quad \dots \\ u_{j+1} &= w_{j,x} + F_{j+1}(w_1, w_2, \dots, w_{j-1}), \quad \dots, \end{aligned}$$

where F_{j+1} are differential polynomials of weight $j + 1$ (note $\text{wt}(w_j) = j$). Thus the w_j can be considered as the primary variables whose x -derivatives determine the KP variables as illustrated by the KP and pKP equations mentioned in Section 2.1. The evolutions of the w_j with respect to the time variables t_n can be prescribed in a consistent fashion by requiring that the dressing operator W satisfy the following system of equations,

$$(3.7) \quad \partial_{t_n} W = B_n W - W \partial^n \quad \text{for } n = 1, 2, \dots,$$

where B_n is now given by $B_n = (W\partial^n W^{-1})_{\geq 0}$. Notice that this expression for B_n as a differential operator is a consequence of the equations, $[(\partial_{t_n} W)W^{-1}]_{\geq 0} = 0$. Equation (3.7) is sometimes referred to as the Sato equation.

The following Theorem asserts that the KP hierarchy is generated by the dressing (or conjugation) of the trivial commutation relation $[\partial_{t_n} - \partial^n, \partial] = 0$ by the operator W . Because of this result, the (inverse) gauge transformation, $\partial \rightarrow L$, is called the dressing transformation for the KP hierarchy.

Theorem 3.2. *If the operator W satisfies the Sato equation (3.7), then the operator $L = W\partial W^{-1}$ satisfies the Lax equation (3.2) for the KP hierarchy, and the operators $B_n = (W\partial^n W^{-1})_{\geq 0} = (L^n)_{\geq 0}$ satisfies the Zakharov-Shabat equations (3.3).*

Proof. First, a direct calculation using $L = W\partial W^{-1}$ and $L^n = W\partial^n W^{-1} = B_n + (L^n)_{<0}$ shows that the operator

$$\begin{aligned} W(\partial_{t_n} - \partial^n)W^{-1} &= \partial_{t_n} - (\partial_{t_n} W)W^{-1} - W\partial^n W^{-1} \\ &= \partial_{t_n} - ((\partial_{t_n} W) + W\partial^n)W^{-1} = \partial_{t_n} - B_n, \end{aligned}$$

where the last equality is due to (3.7). Then equations (3.2) and (3.3) follow from the commutator relations

$$\begin{aligned} 0 &= W[\partial_{t_n} - \partial^n, \partial]W^{-1} = \partial_{t_n} L - [B_n, L], \\ 0 &= W[\partial_{t_n} - \partial^n, \partial_{t_m} - \partial^m]W^{-1} = [\partial_{t_n} - B_n, \partial_{t_m} - B_m]. \end{aligned}$$

□

It follows from Theorem 3.2 that the flows defined by the Sato equation (3.7) are commutative, i.e. they satisfy the compatibility condition $\partial_{t_m}(\partial_{t_n} W) = \partial_{t_n}(\partial_{t_m} W)$. Indeed, the compatibility condition for (3.7) is equivalent to $[\partial_{t_n} + (L^n)_{<0}, \partial_{t_m} + (L^m)_{<0}] = 0$. Using $L^n = B_n + (L^n)_{<0}$, the commutator term on the left hand side can be decomposed as $[\partial_{t_n} - B_n, \partial_{t_m} - B_m] + [\partial_{t_n} - B_n, L^m] - [\partial_{t_m} - B_m, L^n]$, which vanish due to Theorem (3.2). Finally we note that the KP linear system (3.4) is obtained by the dressing action: $\phi = W\phi_0$ where the vacuum eigenfunction ϕ_0 satisfies the *bare* linear system

$$(3.8) \quad \begin{cases} \partial\phi_0 &= k\phi_0, \\ \partial_{t_n}\phi_0 &= \partial^n\phi_0 = k^n\phi_0, \quad n = 1, 2, \dots \end{cases}$$

This equation together with the Sato equation (3.7) form the basic ingredients of the dressing procedure. We will use the bare eigenfunction in the normalized form,

$$(3.9) \quad \phi_0(\mathbf{t}; k) = e^{\theta(\mathbf{t}; k)} \quad \text{with} \quad \theta(\mathbf{t}; k) = \sum_{n=1}^{\infty} k^n t_n.$$

3.3. The τ -function. The τ -function (2.14) introduced in Section 2.3 plays a fundamental role in Sato theory for the KP hierarchy. In this subsection we demonstrate explicitly how the τ -function is related to the dressing operator W which satisfies the Sato equation (3.7). We restrict our discussion in this paper to a finite truncation of the infinite order pseudo-differential operator W for simplicity only, while capturing the flavor of the general theory. Let us then consider the dressing operator

$$W = 1 - w_1\partial^{-1} - w_2\partial^{-2} - \dots - w_N\partial^{-N},$$

and define the differential operator,

$$W_N := W\partial^N = \partial^N - w_1\partial^{N-1} - w_2\partial^{N-2} - \dots - w_N.$$

Since W satisfies the Sato equation, it is immediate that W_N also satisfies

$$\frac{\partial W_N}{\partial t_n} = B_n W_N - W_N \partial^n \quad B_n = (W\partial^n W^{-1})_{\geq 0}.$$

The following Proposition establishes the compatibility conditions leading to the extended Burgers hierarchy introduced in Section 2.

Proposition 3.1. *The N -th order differential equation $W_N f = 0$ is invariant under any flow of the linear heat hierarchy (2.3), $\{\partial_{t_n} f = \partial_x^n f : n = 1, 2, \dots\}$.*

Proof. It suffices to show that $\partial_{t_n}(W_N f) = 0$. Then the desired result follows from the uniqueness of the differential equation:

$$\begin{aligned} \partial_{t_n}(W_N f) &= \partial_{t_n}(W_N) f + W_N \partial_{t_n} f \\ &= (B_n W_N - W_N \partial^n) f + W_N \partial_{t_n} f \\ &= B_n(W_N f) + W_N(\partial_{t_n} f - \partial_x^n f) = 0. \end{aligned}$$

□

Proposition 3.1 provides the compatible system considered in Section 2.3

$$(3.10) \quad \begin{cases} W_N f = 0, \\ \partial_{t_n} f = \partial_x^n f, \quad n = 1, 2, \dots \end{cases}$$

Furthermore, a set $\{f_j : j = 1, 2, \dots, N\}$ of linearly independent solutions of $W_N f = 0$, i.e.

$$f^{(N)} = w_1 f^{(N-1)} + w_2 f^{(N-2)} + \dots + w_{N-1} f^{(1)} + w_N f,$$

can be employed to explicitly construct the coefficient functions w_i of the dressing operator W as follows

$$(3.11) \quad w_i = \frac{(-1)^{i+1}}{\tau} \begin{vmatrix} f_1 & \dots & f_1^{(N-i-1)} & f_1^{(N-i+1)} & \dots & f_1^{(N)} \\ f_2 & \dots & f_2^{(N-i-1)} & f_2^{(N-i+1)} & \dots & f_2^{(N)} \\ \vdots & & \vdots & \vdots & & \vdots \\ f_N & \dots & f_N^{(N-i-1)} & f_N^{(N-i+1)} & \dots & f_N^{(N)} \end{vmatrix},$$

where $\tau = \text{Wr}(f_1, \dots, f_N)$. Note that since the t_n -dependence of the w_i is given via the evolution equations $\partial_{t_n} f_j = \partial_x^n f_j$ for $j = 1, 2, \dots, N$, one also has an explicit solution of the Sato equation (3.7).

The formula (3.11) for the coefficients w_i can be exploited to obtain an elegant expression for the KP eigenfunction ϕ via the τ -function (see for example [8, 24]).

Lemma 3.1. *The eigenfunction ϕ of the linear system (3.4) can be expressed as*

$$\phi(\mathbf{t}; k) = \frac{\tau(\mathbf{t} - [k^{-1}])}{\tau(\mathbf{t})} \phi_0(\mathbf{t}; k),$$

where $\phi_0 = e^{\theta(\mathbf{t}; k)}$ with $\theta(\mathbf{t}; k) = \sum_{n=1}^{\infty} k^n t_n$, and

$$(\mathbf{t} - [k^{-1}]) := \left(t_1 - \frac{1}{k}, t_2 - \frac{1}{2k^2}, t_3 - \frac{1}{3k^3}, \dots \right).$$

We provide a simple proof of this well-known formula to preserve the self-contained style of the present article (see also [8]).

Proof. First note that using (3.11), the eigenfunction ϕ can be written as

$$\begin{aligned} \phi &= W \phi_0 = \left(1 - \frac{w_1}{k} - \frac{w_2}{k^2} - \dots - \frac{w_N}{k^N} \right) \phi_0 \\ &= \frac{1}{\tau} \begin{vmatrix} f_1 & f_1^{(1)} & \dots & f_1^{(N)} \\ \vdots & \vdots & \ddots & \vdots \\ f_N & f_N^{(1)} & \dots & f_N^{(N)} \\ k^{-N} & k^{-N+1} & \dots & 1 \end{vmatrix}. \end{aligned}$$

Using the elementary column operations, the determinant in the numerator of the above expression can be re-written as

$$\frac{(-1)^N}{k^N} \left| \left(f_i^{(j)} - k f_i^{(j-1)} \right)_{1 \leq i, j \leq N} \right|.$$

From the integral representation of the functions f_i ,

$$f_i(\mathbf{t}) = \int_C e^{\theta(\mathbf{t}; \lambda)} \rho_i(\lambda) d\lambda \quad \text{for } i = 1, 2, \dots, N,$$

each matrix element in this determinant is given by

$$\begin{aligned}
f_i^{(j)}(\mathbf{t}) - k f_i^{(j-1)}(\mathbf{t}) &= -k \int_C \lambda^{j-1} \left(1 - \frac{\lambda}{k}\right) e^{\theta(\mathbf{t}; \lambda)} \rho_i(\lambda) d\lambda \\
&= -k \int_C \lambda^{j-1} e^{-\sum_{n=1}^{\infty} \frac{\lambda^n}{n k^n}} e^{\theta(\mathbf{t}; \lambda)} \rho_i(\lambda) d\lambda \\
&= -k e^{-\sum_{n=1}^{\infty} \frac{1}{n k^n} \partial_{t_n}} f_i^{(j-1)}(\mathbf{t}) = -k f_i^{(j-1)}(\mathbf{t} - [k^{-1}]),
\end{aligned}$$

where we have used $\ln(1 - \frac{\lambda}{k}) = -\sum_{n=1}^{\infty} \frac{\lambda^n}{n k^n}$. This completes the proof. \square

One should note that the expression for ϕ in Lemma 3.1 also holds in the general case for the full untruncated version of the operator W . Expanding this formula with respect to k , we have an explicit formula for w_i 's in terms of the τ -function, i.e.

$$w_i = -\frac{1}{\tau} S_i(-\tilde{\partial})\tau,$$

where $\tilde{\partial} := (\partial_{t_1}, \frac{1}{2}\partial_{t_2}, \frac{1}{3}\partial_{t_3}, \dots)$ and $S_n(\mathbf{z})$'s are the complete homogeneous symmetric functions (or sometimes referred to as the elementary Schur polynomials) defined by

$$(3.12) \quad \exp\left(\sum_{n=1}^{\infty} k^n z_n\right) = \sum_{n=0}^{\infty} S_n(\mathbf{z}) k^n, \quad \mathbf{z} = (z_1, z_2, \dots).$$

The explicit expressions of these polynomials are given by

$$\begin{aligned}
S_0(\mathbf{z}) &= 1, \quad S_1(\mathbf{z}) = z_1, \quad S_2(\mathbf{z}) = z_2 + \frac{z_1^2}{2}, \quad S_3(\mathbf{z}) = z_3 + z_1 z_2 + \frac{z_1^3}{6}, \quad \dots, \\
S_n(\mathbf{z}) &= \sum_{l_1 + 2l_2 + \dots + n l_n = n} \frac{z_1^{l_1} z_2^{l_2} \dots z_n^{l_n}}{l_1! l_2! \dots l_n!}.
\end{aligned}$$

Note that for an N -truncated operator W (i.e. $w_n = 0$ if $n > N$) the τ -function satisfies the constraints

$$S_n(-\tilde{\partial})\tau = 0 \quad \text{for } n > N.$$

Lemma 3.1 will be now used to derive a set of first integrals of the KP equation that will prove to be useful in our classification of the line-soliton solutions in Section 5. The integrability of the KP equation may be demonstrated by the existence of an infinite number of conservation laws in the form,

$$\partial_{t_n} h_j = \partial_x g_{j,n},$$

for some conserved densities h_j and the corresponding conserved fluxes $g_{j,n}$. These functions are differential polynomials of u_i 's in the Lax operator L , and they can be found as follows: Differentiating the quantity $\phi^{-1} \partial_x \phi$ with respect to t_n and using the evolution equation $\partial_{t_n} \phi = B_n \phi$, we first derive the conservation law,

$$(3.13) \quad \partial_{t_n} (\phi^{-1} \partial_x \phi) = \partial_x (\phi^{-1} B_n \phi).$$

Next we invert (3.1) using the generalized Leibnitz rule to express the differential operator ∂ in $\partial_x \phi$ in terms of L as

$$\partial = L - v_2 L^{-1} - v_3 L^{-2} - \dots,$$

where v_j 's are the *differential polynomials* of u_i 's. Then the conserved density $\phi^{-1} \partial_x \phi$ can be written as an infinite series after using $L^n \phi = k^n \phi$, $n \in \mathbb{Z}$,

$$\phi^{-1} \partial_x \phi = k - \frac{v_2}{k} - \frac{v_3}{k^2} - \dots,$$

Each function v_j is a conserved density of the KP hierarchy; the first few are given by

$$v_2 = u_2, \quad v_3 = u_3, \quad v_4 = u_4 + u_2^2, \quad v_5 = u_5 - 3u_2u_3 + u_2u_{2,x}, \quad \dots$$

Note that these are nonlocal in the KP solution, $u = 2u_2$ (except v_2). However, they can be expressed simply as certain differential polynomials of the variable $w_1 = \partial_x(\ln \tau)$. Namely, we have

$$\begin{aligned} \phi^{-1} \partial_x \phi &= \partial_x \ln \phi = \partial_x [\theta(\mathbf{t}; k) + \ln \tau(\mathbf{t} - [k^{-1}]) - \ln \tau(\mathbf{t})] \\ &= k + \partial_x \left[\exp \left(- \sum_{n=1}^{\infty} \frac{1}{nk^n} \partial_{t_n} \right) - 1 \right] \ln \tau(\mathbf{t}) = k + \sum_{n=1}^{\infty} \frac{1}{k^n} \partial_x S_n(-\tilde{\partial}) \ln \tau, \end{aligned}$$

which leads to

$$(3.14) \quad v_{n+1} = -\partial_x S_n(-\tilde{\partial}) \ln \tau = -S_n(-\tilde{\partial}) w_1.$$

For example, we have

$$\begin{aligned} v_2 &= \partial_x^2 \ln \tau, \quad v_3 = \frac{1}{2} \partial_x (\partial_{t_2} - \partial_x^2) \ln \tau, \quad v_4 = \frac{1}{3} \partial_x (\partial_{t_3} - \frac{3}{2} \partial_x \partial_{t_2} + \frac{1}{2} \partial_x^3) \ln \tau, \dots \\ v_{n+1} &= \frac{1}{n} \partial_x (\partial_{t_n} + (\text{h.o.d.})) \ln \tau, \quad \dots \end{aligned}$$

where (h.o.d) indicates the terms including higher powers of the derivatives. Moreover, if the solutions u_i 's of the KP hierarchy decrease rapidly to zero as $|x| \rightarrow \infty$, one can define the integrals C_n by

$$(3.15) \quad C_n := \int_{-\infty}^{\infty} v_{n+1}(x, \dots) dx \quad \text{for } n = 1, 2, \dots$$

In particular, if the τ -function gives one line-soliton of $[i, j]$ -type, i.e. $\tau(\mathbf{t}) = E_i(\mathbf{t}) + aE_j(\mathbf{t})$ with $E_i(\mathbf{t}) = \exp(\theta(\mathbf{t}; k_i))$, then from (3.14) the integrals take

$$(3.16) \quad C_n = -S_n(-\tilde{\partial}) \ln \tau \Big|_{x=-\infty}^{x=\infty} = \frac{1}{n} (k_j^n - k_i^n).$$

Notice here that if $k_i < k_j$, $\tau \approx E_i = e^{\theta(\mathbf{t}; k_i)}$ for $x \ll 0$ and $\tau \approx aE_j = ae^{\theta(\mathbf{t}; k_j)}$ for $x \gg 0$. In general, if there are N line-solitons of $[i_l, j_l]$ -type for $l = 1, \dots, N$, then we have

$$C_n = \sum_{l=1}^N \frac{1}{n} (k_{j_l}^n - k_{i_l}^n),$$

(see Corollary 5.1 below for the proof). This expression is similar to the N -soliton solutions of the KdV equation (see for example [36]), but for the KP equation, the C_n 's are also independent of y . This fact will be used in Proposition 5.1 below.

Remark 3.3. An alternative set of the conserved densities can be found by observing the following tautological equations (see p.99 in [8]),

$$\partial_{t_m} (\partial_x \partial_{t_n} \ln \tau) = \partial_x (\partial_{t_m} \partial_{t_n} \ln \tau).$$

That is, the conserved densities obtained from those equations are given by

$$\tilde{v}_{n+1} := \frac{1}{n} \partial_x \partial_{t_n} \ln \tau.$$

However one can show that the integrals $\tilde{C}_n = \int_{-\infty}^{\infty} \tilde{v}_{n+1} dx = \int_{-\infty}^{\infty} v_{n+1} = C_n$ for all n .

In Section 5, we will discuss the classification problem of the solutions obtained from the τ -function with finite dimensional solutions of the f -equation, and will show that the classification is completely characterized by the asymptotic behavior of the τ -function. Before getting into the classification problem, in the next Section, we give a brief review on the real Grassmann manifold for the background information necessary for the problem.

4. INTRODUCTION TO THE GRASSMANNIAN $\text{Gr}(N, M)$

The main purpose of this Section is to explain a mathematical background of soliton solutions of the KP equation which is provided by the real Grassmannian $\text{Gr}(N, M)$, the set of N -dimensional subspaces of \mathbb{R}^M . This is a finite dimensional version of the universal Grassmannian introduced by Sato in 1981 [31].

4.1. Grassmannian $\text{Gr}(N, M)$. Let $\{f_1, f_2, \dots, f_N\}$ be a set of N linearly independent vectors in \mathbb{R}^M . Then the set forms a basis for V ,

$$V = \text{Span}_{\mathbb{R}}\{f_1, f_2, \dots, f_N\}, \quad \text{and} \quad \text{Gr}(N, M) = \{V : V \subset \mathbb{R}^M\}.$$

Let us begin with some simple examples:

Example 4.1. The Grassmannian $\text{Gr}(1, M)$: Suppose $M = 2$, then $\text{Gr}(1, 2)$ is the set of all lines through the origin in the plane. Thus, any two non-zero vectors in \mathbb{R}^2 are on the same line if they are proportional, and they determine the *same* point of $\text{Gr}(1, 2)$. Since every point $(x_1, x_2) \neq (0, 0)$ of \mathbb{R}^2 lies on some line l through the origin, we can represent l , which is also a *point* on $\text{Gr}(1, 2)$, as

$$\text{Gr}(1, 2) \ni l = (x_1 : x_2) := \{c(x_1, x_2) \mid c \neq 0, (x_1, x_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}\}.$$

The line $(x_1 : x_2)$ is called the homogeneous coordinate for $\text{Gr}(1, 2)$ which has then the structure of a one-dimensional manifold. This manifold admits the decomposition,

$$\text{Gr}(1, 2) = \{(1 : a) : a \in \mathbb{R}\} \cup \{(0 : 1)\} \cong \mathbb{R} \cup \{\infty\},$$

which is the projective line $\mathbb{R}P^1$, and it can be identified with the unit circle S^1 (i.e. $a \in \mathbb{R}$ in the decomposition parametrizes the slope of each line, and the y -axis is identified as $(0 : 1)$).

The set of all lines passing through the origin of \mathbb{R}^M is represented by the Grassmannian $\text{Gr}(1, M)$ which can be identified with the $M - 1$ dimensional projective space $\mathbb{R}P^{M-1}$. Homogeneous coordinates and the decomposition of $\text{Gr}(1, M)$ can be described in the same way as in the previous example. In this case, one has the well-known Schubert decomposition given by

$$\text{Gr}(1, M) = \bigcup_{j=1}^M F_j \quad \text{with} \quad F_j = \{(0 : \dots : 0 : 1 : a_1 : \dots : a_{M-j}) : a_k \in \mathbb{R}\},$$

where each *cell* F_j has the structure of a $(M - j)$ -dimensional vector space; the last one being $F_M = (0 : \dots : 0 : 1)$ which represents a point (the 0-dimensional cell) in $\text{Gr}(1, M)$.

We now consider a representation for the general case of $\text{Gr}(N, M)$ which plays a key role in the classification of the line-soliton solutions of the KP equation as we show in the next section. With respect to a basis $\{E_1, E_2, \dots, E_M\}$ of \mathbb{R}^M , the basis vectors f_i of the N -dimensional subspace V can be expressed as

$$(4.1) \quad f_i = \sum_{j=1}^M a_{ij} E_j, \quad \text{for } i = 1, 2, \dots, N,$$

where $A := (a_{ij})$ is a $N \times M$ matrix. Note that the rank of A is N since the $\{f_1, f_2, \dots, f_N\}$ are linearly independent. A different choice of basis for V will induce row operations on the original matrix: $A \rightarrow BA$ with $B \in \text{GL}(N)$. Consequently, any matrix A can be canonically chosen in the *reduced row echelon form* (RREF) by taking appropriate $B \in \text{GL}(N)$. This canonical form of A has a distinguished set of *pivot* columns labeled by $\mathcal{I} = \{i_1, i_2, \dots, i_N\}$ with $1 \leq i_1 < i_2 < \dots < i_N \leq M$ such that the submatrix $A_{\mathcal{I}}$ formed by the column set \mathcal{I} is the identity matrix I_N . For any given A , its unique RREF provides a coordinate for a point of $\text{Gr}(N, M)$. Moreover, the set of all such A matrices in RREF, and with a fixed pivot set \mathcal{I} forms an affine coordinate chart or *cell* $W_{\mathcal{I}}$ which (similar to $\text{Gr}(1, M)$) gives the Schubert decomposition of the Grassmannian

$$(4.2) \quad \text{Gr}(N, M) = \bigcup_{\mathcal{I} \subset \{1, \dots, M\}} W_{\mathcal{I}}.$$

Each cell $W_{\mathcal{I}}$ is called a Schubert cell. For example, if $\mathcal{I} = \{1, 2, \dots, N\}$, then the Schubert cell $W_{\mathcal{I}}$ contains all A matrices whose RREF is given by

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & * & \cdots & * \\ 0 & 1 & \cdots & 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & * & \cdots & * \end{pmatrix}$$

where the $N(M - N)$ entries of the right-hand block are arbitrary. This particular Schubert cell $W_{\mathcal{I}}$ is often referred to as the *top* cell of the Grassmannian $\text{Gr}(N, M)$, and has the maximum number of free parameters marked by $*$. It follows from this that the dimension of $\text{Gr}(N, M)$ is $N(M - N)$. A straightforward counting of the maximum number of free parameters for an A in RREF with a given pivot set $\mathcal{I} = \{i_1, \dots, i_N\}$ shows that the dimension of the cell $W_{\mathcal{I}}$ is given by

$$\dim W_{\mathcal{I}} = N(M - N) + \frac{1}{2}N(N + 1) - (i_1 + i_2 + \cdots + i_N).$$

Example 4.2. For $\text{Gr}(2, 4)$, the Schubert decomposition (4.2) is given by

$$\text{Gr}(2, 4) = \bigcup_{1 \leq i < j \leq 4} W_{\{i, j\}}.$$

There are six cells $W_{\{i, j\}}$ with $\dim W_{\{i, j\}} = 7 - (i + j)$, which are given by

$$\begin{aligned} \text{(a)} \quad W_{\{1, 2\}} &= \left\{ \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix} \right\} & \text{(b)} \quad W_{\{1, 3\}} &= \left\{ \begin{pmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix} \right\} \\ \text{(c)} \quad W_{\{1, 4\}} &= \left\{ \begin{pmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} & \text{(d)} \quad W_{\{2, 3\}} &= \left\{ \begin{pmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix} \right\} \\ \text{(e)} \quad W_{\{2, 4\}} &= \left\{ \begin{pmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} & \text{(f)} \quad W_{\{3, 4\}} &= \left\{ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \end{aligned}$$

4.2. The Plücker embedding and the τ -function. We now introduce a coordinate system called the Plücker coordinates for the Grassmannian $\text{Gr}(N, M)$ which is related directly to the Wronskian form of the τ -function (2.14) in Theorem 2.5. Given any basis $\{f_1, \dots, f_N\}$ of an N -dimensional subspace $V \subset \mathbb{R}^M$, the exterior product space $\bigwedge^N V$ is a one-dimensional subspace $\bigwedge^N V \subset \bigwedge^N \mathbb{R}^M$ spanned by the wedge product $\nu_f := f_1 \wedge f_2 \wedge \cdots \wedge f_N$. If $\{g_1, \dots, g_N\}$ is another basis for V with $B \in \text{GL}(N)$ so that $(g_1, \dots, g_N) = (f_1, \dots, f_N)B$, then $\nu_g = \det(B)\nu_f$. Consequently, $\bigwedge^N V$ is an element of $P(\bigwedge^N \mathbb{R}^M)$, the projectivization of $\bigwedge^N \mathbb{R}^M$. The Plücker embedding is a map from $\text{Gr}(N, M)$ to $P(\bigwedge^N \mathbb{R}^M)$, that sends $V \subset \mathbb{R}^M$ to $\bigwedge^N V \subset \bigwedge^N \mathbb{R}^M$. In terms of the A -matrix in (4.1), the explicit form of this map is given by

$$(4.3) \quad \nu_f = f_1 \wedge f_2 \wedge \cdots \wedge f_N = \sum_{1 \leq i_1 < \cdots < i_N \leq M} \xi(i_1, i_2, \dots, i_N) E_{i_1} \wedge E_{i_2} \wedge \cdots \wedge E_{i_N},$$

where $\xi(i_1, i_2, \dots, i_N)$ are the maximal minors of A formed by its columns A_i , $i \in \{i_1, i_2, \dots, i_N\}$. Recall that the maximal minors were introduced in Section 1.3 (following Theorem 2.5). The minors in the set $\{\xi(i_1, \dots, i_N) : 1 \leq i_1 < \cdots < i_N \leq M\}$ are defined up to a scale factor, and are called the Plücker coordinates for the Grassmannian $\text{Gr}(N, M)$. These coordinates are not independent, but satisfy the identities

$$(4.4) \quad \sum_{i=1}^{N+1} (-1)^{i-1} \xi(\alpha_1, \dots, \alpha_{N-1}, \beta_i) \xi(\beta_1, \dots, \beta_{i-1}, \beta_{i+1}, \dots, \beta_{N+1}) = 0,$$

called the Plücker relations which hold because the exterior form $\nu_f = f_1 \wedge \cdots \wedge f_N$ is totally decomposable, i.e., $v \wedge \nu_f = 0$, $\forall v \in V$. The Plücker relations can be derived using elementary linear

algebra from the Laplace expansion of the following $2N \times 2N$ determinant formed by the columns A_i of the matrix A , similar to Lemma 2.1,

$$\begin{vmatrix} A_{\alpha_1} & \cdots & A_{\alpha_{N-1}} & A_{\beta_1} & \cdots & A_{\beta_{N+1}} \\ 0 & \cdots & 0 & A_{\beta_1} & \cdots & A_{\beta_{N+1}} \end{vmatrix} = 0.$$

The Plücker coordinates, modulo the Plücker relations, give the correct dimension of $\text{Gr}(N, M)$ which is typically less than the dimension of $P(\bigwedge^N \mathbb{R}^M)$.

Example 4.3. For $\text{Gr}(2, 4)$, the Plücker coordinates are given by the maximal minors,

$$\xi(1, 2), \quad \xi(1, 3), \quad \xi(1, 4), \quad \xi(2, 3), \quad \xi(2, 4), \quad \xi(3, 4).$$

Taking $\alpha_1 = 1$, $(\beta_1, \beta_2, \beta_3) = (2, 3, 4)$ in (4.4) gives the only Plücker relation in this case,

$$\xi(1, 2)\xi(3, 4) - \xi(1, 3)\xi(2, 4) + \xi(1, 4)\xi(2, 3) = 0,$$

which is the same as (2.15). Since $\dim(\bigwedge^2 \mathbb{R}^4) = 6$, and the projectivization gives $\dim(P(\bigwedge^2 \mathbb{R}^4)) = 6 - 1 = 5$. Then with one Plücker relation, the dimension of $\text{Gr}(2, 4)$ turns out to be 4, which is consistent with the dimension of the top cell $W_{\{1,2\}}$ as shown in Example 3.2 (case (a)).

We next show that the τ -function defined by the Wronskian determinant in (2.14) can be identified as a point on the Grassmannian $\text{Gr}(N, M)$. First note that the set of functions,

$$\left\{ E_j = e^{\theta_j} = \exp \left(\sum_{n=1}^{\infty} k_j^n t_n \right) : j = 1, 2, \dots, M \right\},$$

with distinct real parameters k_j , gives a linearly independent set because $\text{Wr}(E_1, E_2, \dots, E_M) \neq 0$. Then an N -dimensional subspace $V \subset \mathbb{R}^M$ is given by $\text{Span}_{\mathbb{R}}\{f_1, f_2, \dots, f_N\}$ with N linearly independent solutions $\{f_1, f_2, \dots, f_N\}$ of $f_y = f_{xx}$ and $f_t = f_{xxx}$ as in Theorem 2.5. The coefficients of E_j in (4.1) form a $N \times M$ matrix A of rank N , whose row-space is isomorphic to V so that A represents a point of $\text{Gr}(N, M)$. One can then see that the τ -function $\tau = \text{Wr}(f_1, f_2, \dots, f_N)$ gives the Plücker embedding $\text{Gr}(N, M) \rightarrow P(\bigwedge^N \mathbb{R}^M)$ defined in (4.3): Indeed expanding the Wronskian determinant (2.14) by Binet-Cauchy formula, we have

$$(4.5) \quad \tau = \text{Wr}(f_1, f_2, \dots, f_N) = \sum_{1 \leq i_1 < \dots < i_N \leq M} \xi(i_1, i_2, \dots, i_N) E(i_1, i_2, \dots, i_N),$$

where $\xi(i_1, i_2, \dots, i_N)$ are the Plücker coordinates given by the maximal minors of the A -matrix, and $E(i_1, i_2, \dots, i_N) = \text{Wr}(E_{i_1}, E_{i_2}, \dots, E_{i_N})$. Note here that $E(i_1, i_2, \dots, i_N)$ can be identified as $E_{i_1} \wedge E_{i_2} \wedge \dots \wedge E_{i_N}$, and forms a basis for $\bigwedge^N \mathbb{R}^M$, i.e.

$$\text{Span}_{\mathbb{R}} \{E(i_1, i_2, \dots, i_N) : 1 \leq i_1 < i_2 < \dots < i_N \leq M\} \cong \bigwedge^N \mathbb{R}^M.$$

Note here that the sum $k_{i_1} + k_{i_2} + \dots + k_{i_N}$ should be distinct for distinct sets $\{i_1, i_2, \dots, i_N\}$ in order for the $\{E(i_1, i_2, \dots, i_N)\}$ to be independent.

Remark 4.4. We just mention here that an infinite dimensional Grassmann manifold appears when we consider the case where the functions $\{f_1, \dots, f_N\}$ are analytic. In this case, one can expand $f_i(\mathbf{t})$ in terms of the elementary Schur polynomials defined in (3.12),

$$f_i(\mathbf{t}) = \int_C e^{\theta(\mathbf{t}; k)} \rho_i(k) dk = \sum_{n=1}^{\infty} a_{i,n} S_{n-1}(\mathbf{t}),$$

where $\theta(\mathbf{t}; k) = \sum_{n=1}^{\infty} k^n t_n$, and $a_{i,n}$ are given by the moment integrals,

$$a_{i,n} = \int_C k^n \rho_i(k) dk \quad \text{for} \quad \begin{cases} 1 \leq i \leq N, \\ 1 \leq n. \end{cases}$$

Thus the A -matrix has the size $N \times \infty$, and each A -matrix represents a point on an infinite dimensional Grassmann manifold $\text{Gr}(N, \infty)$. The τ -function then has the well-known expansion form with the Schur polynomials,

$$(4.6) \quad \tau(\mathbf{t}) = \sum_{1 \leq i_1 < i_2 < \dots < i_N} \xi(i_1, i_2, \dots, i_N) S_{\{i_1, i_2, \dots, i_N\}}(\mathbf{t}),$$

where $S_{\{i_1, \dots, i_N\}}(\mathbf{t})$ is defined by

$$S_{\{i_1, i_2, \dots, i_N\}}(\mathbf{t}) = \text{Wr}(S_{i_1-1}, S_{i_2-1}, \dots, S_{i_N-1})(\mathbf{t}).$$

The index set $\{i_1, i_2, \dots, i_N\}$ is commonly expressed by a Young diagram (see for example [24]). Then the main result of the Sato theory is that the function defined by (4.6) is the τ -function, which represents a point on the Grassmann manifold [31].

In the following section, we will show that the line-soliton solutions arise from the τ -function given by (4.5). The Grassmannian formulation of this τ -function provides a geometric characterization of the solutions by identifying the different solution classes with distinct Grassmann cells. Of particular physical interests are those solutions which are regular in x, y and t . Such solutions are realized if the associated τ -function has non-negative Plücker coordinates, that is, all $\xi(i_1, i_2, \dots, i_N) \geq 0$ in (4.5). The matrices A having this property are called *totally non-negative* (TNN) matrices; they parametrize the totally non-negative Grassmannian $\text{Gr}^+(N, M) \subset \text{Gr}(N, M)$ (see also [30]).

5. CLASSIFICATION OF SOLITON SOLUTIONS

In this section, we present a classification scheme of the line-soliton solutions based on the asymptotic behavior of the τ -function (4.5). It turns out that the classification problem is closely related to the characterization of the totally nonnegative Grassmannian cells of $\text{Gr}(N, M)$ recently studied by Postnikov et al (see for example [30, 35]).

5.1. Asymptotic line-solitons. Let us recall that the τ -function is given by the sum of exponential terms,

$$(5.1) \quad \tau(\mathbf{t}) = \sum_{1 \leq m_1 < \dots < m_N \leq M} \xi(m_1, \dots, m_N) E(m_1, \dots, m_N)(\mathbf{t}),$$

where $E(m_1, \dots, m_N) = \text{Wr}(E_{m_1}, \dots, E_{m_N})$ with $E_m(\mathbf{t}) = e^{\theta(\mathbf{t}; k_m)} = \exp(\sum_{n=1}^{\infty} k_m^n t_n)$. We then impose the following conditions:

- (i) The k -parameters are ordered as $k_1 < k_2 < \dots < k_M$, and the sums $k_i + k_j$ are all distinct (this implies that all $[i, j]$ -solitons have different slopes).
- (ii) The A -matrix is TNN, that is, all its maximal minors $\xi(m_1, m_2, \dots, m_N)$ are non-negative (i.e. the τ -function is positive definite for all x, y, t , hence the solution $u = 2\partial_x^2(\ln \tau)$ is non-singular).

The asymptotic spatial structure of the solution $u(x, y, t)$ is determined from the consideration of which exponential term $E(m_1, \dots, m_N)$ in the τ -function dominates in different regions of the xy -plane for large $|y|$. For example, if only one exponential term $E(m_1, \dots, m_N)$ in the τ -function is dominant in a certain region, then the solution $u = 2\partial_x^2(\ln \tau)$ remains exponentially small at all points in the interior of any given dominant region, but is localized at the *boundaries* of two distinct regions where a balance exists between two dominant exponentials in the τ -function (5.1).

Furthermore, it is also possible to deduce the following result from the asymptotic analysis of the τ -function:

Proposition 5.1. *The dominant exponentials of the τ -function in adjacent regions of the xy -plane are of the form $E(i, m_2, \dots, m_N)$ and $E(j, m_2, \dots, m_N)$. That is, the exponentials contain $N - 1$ common phases and differ by only one phase.*

We prove the Proposition by applying the following Lemma.

Lemma 5.1. *Let $\{s_n(x_1, x_2, \dots, x_N) : n = 1, \dots, N\}$ be the set of N power sums in N real variables defined by*

$$s_n := x_1^n + x_2^n + \dots + x_N^n, \quad n = 1, 2, \dots, N.$$

Suppose the set of N equations

$$s_n(x_1, x_2, \dots, x_N) = c_n, \quad n = 1, \dots, N,$$

admit solutions (x_1, \dots, x_N) for appropriate values of the constants c_n , then the solution set is unique up to the permutation of N elements $\{x_1, \dots, x_N\}$.

Proof. The proof follows from the Newton's identities relating the power sums $\{s_n\}$ and the elementary symmetric polynomials $\{\sigma_n\}$ which are defined as

$$\sigma_n = \sum_{1 \leq i_1 < \dots < i_n \leq N} x_{i_1} x_{i_2} \dots x_{i_n} \quad n = 1, \dots, N.$$

For $n = 1, 2, \dots, N$, the Newton's identities are recursively given by

$$s_n + a_1 s_{n-1} + \dots + a_{n-1} s_1 + n a_n = 0, \quad a_k = (-1)^k \sigma_k.$$

Then the solutions of $s_n = c_n$, $n = 1, \dots, N$, which exist by assumption, correspond the set of N roots x_1, \dots, x_N of the N -th degree monic polynomial,

$$x^N + a_1 x^{N-1} + a_2 x^{N-2} + \dots + a_N = 0,$$

where the coefficients $\{a_n\}$ are uniquely determined by $\{c_n\}$ using Newton's identity. Obviously, the set of roots (x_1, \dots, x_N) is unique up to permutations. \square

Now we prove Proposition 5.1:

Proof. Let $E(i_1, \dots, i_N)$ and $E(j_1, \dots, j_N)$ be the two dominant exponentials in adjacent regions, then near the boundary of the two regions, the τ -function can be approximately given by

$$\tau \approx \xi_1 E(i_1, \dots, i_N) + \xi_2 E(j_1, \dots, j_N),$$

where ξ_1 and ξ_2 are the corresponding minors of the A -matrix. The remaining terms in the τ -function are exponentially small in comparison with those dominant terms. Then the solution $u = 2\partial_x^2(\ln \tau)$ locally has the one-soliton form as in (2.10), i.e.

$$u \approx \mathcal{A} \operatorname{sech}^2 \left(\sum_{n=1}^{\infty} \mathcal{K}_n t_n + \Theta^0 \right),$$

where $\mathcal{A} = \frac{1}{2}(Q - P)^2$ and $\mathcal{K}_n = \frac{1}{2}(Q^n - P^n)$. Moreover, P and Q satisfy

$$Q^n - P^n = (k_{j_1}^n + \dots + k_{j_N}^n) - (k_{i_1}^n + \dots + k_{i_N}^n) \quad n = 1, 2, \dots.$$

Here it is assumed (without any loss of genericity) that $k_{j_1} + \dots + k_{j_N} > k_{i_1} + \dots + k_{i_N}$ so that $P < Q$. Let us rewrite the first $N + 1$ of the above equations as follows

$$(5.2) \quad P^n + k_{j_1}^n + \dots + k_{j_N}^n = Q^n + k_{i_1}^n + \dots + k_{i_N}^n := c_n,$$

where c_n 's are values of each sum. Then $\{P, k_{j_1}, \dots, k_{j_N}\}$ and $\{Q, k_{i_1}, \dots, k_{i_N}\}$ are solutions to the system of equations $x_1^n + \dots + x_{N+1}^n = c_n$, $n = 1, \dots, N + 1$. Hence, from Lemma 5.1 these two sets must be the same up to permutation of their elements. Now recall that the sets $\{k_{i_n}\}$ and $\{k_{j_n}\}$ are distinct and $P \neq Q$. Then the one-to-one correspondence between the sets $\{P, k_{j_1}, \dots, k_{j_N}\}$ and $\{Q, k_{i_1}, \dots, k_{i_N}\}$ implies that P must be one of $\{k_{i_n}\}$, and Q must be one of $\{k_{j_n}\}$. Then the remaining $N - 1$ elements of the set $\{k_{i_n}\}$ must be the same as the remaining $N - 1$ elements of the set $\{k_{j_n}\}$, proving the Proposition. \square

As an immediate consequence of Proposition 5.1, the asymptotic behavior of the KP solution is given by

$$(5.3) \quad u(x, y, t) \approx \frac{1}{2}(k_j - k_i)^2 \operatorname{sech}^2 \frac{1}{2}(\theta_j - \theta_i + \theta_{ij}),$$

in the neighborhood of the line $x + (k_i + k_j)y = \text{constant}$, which forms the boundary between the regions of dominant exponentials $E(i, m_2, \dots, m_N)$ and $E(j, m_2, \dots, m_N)$. Equation (5.3) defines

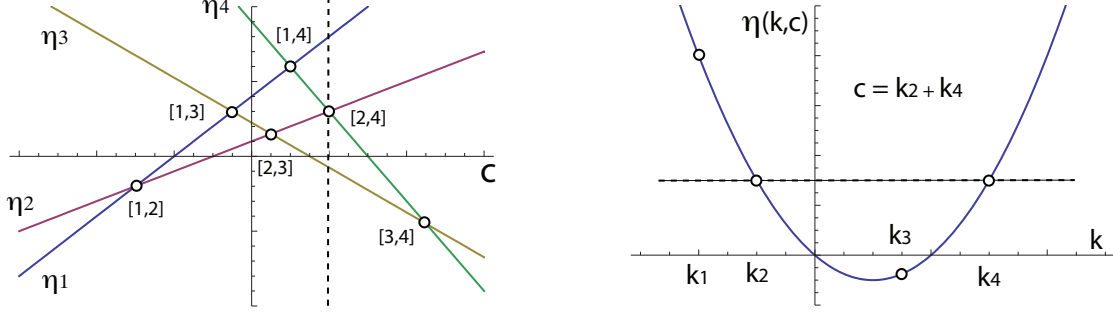


FIGURE 5.1. The left figure shows $\eta_j(c) = k_j(k_j - c)$ for $j = 1, \dots, 4$. Each $[i, j]$ at the intersection point of $\eta_i(c) = \eta_j(c)$ indicates the exchange $i \leftrightarrow j$. We assume that there is at most one intersection point for each c (genericity condition for k_j 's). The right one illustrates the η as the function of k with $c = k_2 + k_3$, that is, the η along the dotted line in the left figure passing through the intersection point $\eta_2(c) = \eta_4(c)$. This figure shows the order $\eta_3 < \eta_2 = \eta_4 < \eta_1$.

an *asymptotic* $[i, j]$ -soliton as a result of those two dominant exponentials. Then the condition (i) below (5.1) implies that those asymptotic solitons have the different slopes, so that they all separate asymptotically for $|y| \gg 0$.

Next we discuss how to determine which exponential term in a given τ -function is actually dominant in the direction of particular line $x = -cy$ for $y \rightarrow \pm\infty$. The basic idea is the same as in Example 2.4. Along the line $x = -cy$, each exponential term $E(m_1, m_2, \dots, m_N)$ has the form,

$$C \exp \left(\sum_{n=1}^N \eta(k_{m_n}, c) y + \theta_{m_n}^0(t) \right),$$

where C and $\theta_{m_n}^0(t)$ are constants for fixed t , and $\eta(k, c)$ is defined by (cf. (2.11)),

$$(5.4) \quad \eta(k, c) = k(k - c) \quad \text{and} \quad \eta_j(c) = k_j(k_j - c).$$

Then for $y \gg 0$ (or $\ll 0$), we look for the dominant (or least) sum of $\eta(k_{m_n}, c)$ for each c . First note that we have

$$\eta_i(c) = \eta_j(c) \quad \text{if} \quad c = k_i + k_j,$$

that is, in the direction of $[i, j]$ -soliton, those terms are in balance. Since the k -parameters are ordered as $k_1 < k_2 < \dots < k_M$, we have the following dominance relation among the other $\eta_m(c)$'s along $c = k_i + k_j$,

$$(5.5) \quad \begin{cases} \eta_i = \eta_j < \eta_m & \text{if } 1 \leq m < i \text{ or } j < m \leq M, \\ \eta_i = \eta_j > \eta_m & \text{if } i < m < j. \end{cases}$$

In order to find the dominant sum, the graph of $\eta(k, c)$ is particularly useful. Figure 5.1 illustrates the case with $M = 4$. We demonstrate how to identify the asymptotic line-solitons from a given τ -function using Lemma 5.1 and the relations (5.5).

Example 5.1. We consider the A -matrix given by

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Then from (5.1) the τ -function has the form,

$$\tau = E(1, 3) + E(1, 4) + E(2, 3) + E(2, 4).$$

Since Proposition 5.1 implies that the line-solitons are localized along the phase transition lines $x + cy = \text{constant}$ with $c = k_i + k_j$, we look for dominant exponential terms in the τ -function along those directions.

For $y \gg 0$, the c -values along a line $x = -cy$ decrease as we sweep clockwise from negative to positive x -axis. So, we start with the largest value $c = k_3 + k_4$, and try to determine if the line $[3, 4]$ actually corresponds to a line-soliton. The relation (5.5) gives $\eta_1, \eta_2 > \eta_3 = \eta_4$, along the line $[3, 4]$. This implies that $\eta_1 + \eta_2$ is the most dominant combinations along this line. However, since the corresponding exponential term $E(1, 2)$ is *absent* in the τ -function, it turns out that $E(1, 3)$ and $E(1, 4)$ are the two dominant exponentials in the τ -function along the line $[3, 4]$ satisfying the condition of Lemma 5.1. Therefore, there is a $[3, 4]$ -soliton. An almost identical argument implies that $[1, 2]$ is another line-soliton. But for example, along the line $[1, 3]$, the following relations hold: $\eta_4 > \eta_1 = \eta_3$, $\eta_2 < \eta_1 = \eta_3$. This implies that $\eta_1 + \eta_4$ and $\eta_3 + \eta_4$ are the two dominant combinations. But $E(1, 4)$ is the only dominant exponential term along the line $[1, 3]$, since $E(3, 4)$ is absent from the τ -function. Hence the solution $u \approx 0$. In this fashion by checking along the lines $[i, j]$ for all possible (i, j) pairs, we conclude that $[1, 2]$ and $[3, 4]$ are the only two asymptotic line-solitons as $y \gg 0$.

For $y \ll 0$, we now look for the least sum $\eta_i + \eta_j$ which implies the dominant exponential $E(i, j)$. We continue to sweep clockwise from positive towards negative x -axis with the c -values still decreasing. Again, we start with $c = k_3 + k_4$, then we have the relation $\eta_1, \eta_2 > \eta_3 = \eta_4$. But since $E(3, 4)$ is absent from the τ -function, $E(2, 3)$ and $E(2, 4)$ provide the dominant balance leading to the line-soliton $[3, 4]$. Considering $c = k_2 + k_4$, we have $\eta_3 < \eta_2 = \eta_1, \eta_4$, which implies that $E(2, 3)$ is the only dominant exponential in the τ -function, hence $u \approx 0$ along the line $[2, 4]$. Proceeding similarly as in the case for $y \gg 0$ leads to the conclusion that $[1, 2]$ and $[3, 4]$ are the only two asymptotic line-solitons for $y \ll 0$.

From the above example, it is important to notice the fact that whether or not the dominant (or least) combinations inferred from (5.5) are actually *present* in the given τ -function ultimately decides if a line $[i, j]$ corresponds to a line-soliton. In turn, this depends on the coefficients of the exponential terms given by the maximal minors of the A -matrix. Therefore, it suffices to specify only the A -matrix rather than the whole τ -function in order to determine the asymptotic line-solitons. The next example illustrates this point.

Example 5.2. Consider the 2×4 matrix,

$$A = \begin{pmatrix} 1 & 0 & 0 & - \\ 0 & 1 & + & + \end{pmatrix},$$

where some of the non-zero entries are indicated by their signs which ensure that all non-zero maximal minors of A are positive. In this case, there are five nonzero minors,

$$\xi(1, 2), \quad \xi(1, 3), \quad \xi(1, 4), \quad \xi(2, 4), \quad \xi(3, 4),$$

and one missing $\xi(2, 3)$.

Now look for the asymptotic solitons as $y \gg 0$. First we can immediately see that $[3, 4]$ -soliton is impossible since the dominant exponents along the line $[3, 4]$ from (5.5) is given by $\eta_1 + \eta_2$. Since $\xi(1, 2) \neq 0$, $\tau(x, y, t) \approx E(1, 2)$ implying that $u \approx 0$ along $[3, 4]$. For the same reason, $[1, 4]$ and $[1, 2]$ -solitons are also impossible. Let us then check the $[2, 4]$ -soliton. From (5.5), $\eta_3 < \eta_2 = \eta_4 < \eta_1$, and since $\xi(1, 2) \neq 0$, $\xi(1, 4) \neq 0$, the τ -function has the dominant phase balance $\tau \approx E(1, 2) + \xi(1, 4)E(1, 4)$ along $[2, 4]$. Therefore $[2, 4]$ corresponds to an asymptotic line-soliton as $y \rightarrow \infty$. Moreover, the $[1, 3]$ -soliton also exists for similar reasons. Thus, we have two asymptotic line-solitons for $y \gg 0$.

We next look for the asymptotic solitons for $y \ll 0$. It is easy to see that $[1, 2]$ and $[3, 4]$ -solitons are impossible. Then consider the $[1, 3]$ -soliton. In this case, (5.5) implies that $\eta_2 < \eta_1 = \eta_3 < \eta_4$. Thus the two exponents $E(1, 2)$ and $E(2, 3)$ are dominant for $y \ll 0$. However, $\xi(2, 3) = 0$ implies that this is impossible. So $[1, 3]$ -soliton does not exist as $y \ll 0$. For similar reasons, $[2, 4]$ -soliton is also

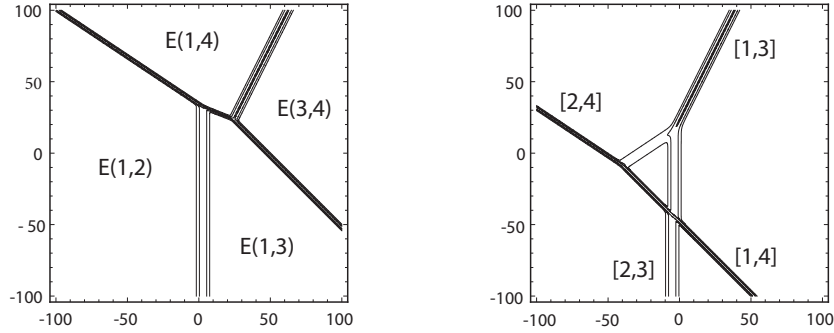


FIGURE 5.2. A $(2,2)$ -soliton solution. The left figure is at $t = -16$ and the right one at $t = 16$. Each $E(i, j)$ for $1 \leq i < j \leq 4$ in the left figure shows the dominant exponential in the region, and the boundaries of those regions give the line-solitons. The parameters (k_1, \dots, k_4) are chosen as $(-1, -\frac{1}{2}, \frac{1}{2}, 2)$.

impossible. Now check $[1, 4]$ -soliton. In this case, we have $\eta_2, \eta_3 < \eta_1 = \eta_4$ from (5.5) (see also Figure 5.1). Since $\xi(2, 3) = 0$, the dominant exponent $E(2, 3)$ is *not* present in the τ -function, but there does exist a dominant balance with $E(1, 3)$ and $E(3, 4)$. Thus there is a dominant phase transition along the line $[1, 4]$. A similar argument applies for the transition line $[2, 3]$ which corresponds the other line-soliton as $y \ll 0$. In Figure 5.2, we show the soliton solution given by this matrix.

5.2. Characterization of the line-solitons. We saw in the Examples 5.1 and 5.2 that the A -matrix plays a key role in the identification of the asymptotic line-solitons since they determine if particular dominant phase combinations are in fact present in the given τ -function. Therefore, in order to obtain a complete characterization of the asymptotic line-solitons it is necessary to consider the structure of the $N \times M$ coefficient matrix A . For the remainder of this article, we will consider the matrix A to be in RREF, and we will also assume that A is *irreducible* as defined below:

Definition 5.3. An $N \times M$ matrix A is irreducible if each column of A contains at least one nonzero element, or each row contains at least one nonzero element other than the pivot once A is in RREF.

The reason for this assumption is the following: if an $N \times M$ matrix A is *not* irreducible, then the solution $u = 2\partial_x^2(\ln \tau)$ can be obtained from a τ -function with a matrix \tilde{A} of smaller size than the original one (that is, the size of A is reducible). If a column of A is identically zero, then it is clear that we can re-express the functions f_i in terms of a $N \times (M - 1)$ coefficient matrix \tilde{A} obtained from A by deleting its zero column. Or, suppose that an $N \times M$ matrix A in RREF has a row whose elements are all zero except for the pivot, then it can be deduced from (4.5) that the corresponding τ -function gives the same KP solution u which can be obtained from another τ -function associated with a $(N - 1) \times (M - 1)$ matrix \tilde{A} . For example, consider the matrix,

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which yields $f_1 = E_1 + a_{12}E_2 + a_{13}E_3$ and $f_2 = E_4$. Then $\tau = \text{Wr}(f_1, f_2) = E_4(k_4f_1 - f_{1,x})$. Factoring out the exponential E_4 , we find that both τ and $\tilde{\tau} = k_4f_1 - f_{1,x}$ give the same KP solution u . Furthermore, note that $\tilde{\tau} = \tilde{f} = (E_1, E_2, E_3)\tilde{A}^T$, where \tilde{A} is the 1×3 matrix,

$$\tilde{A} = (\tilde{a}_{11}, \tilde{a}_{12}, \tilde{a}_{13}) = (k_4 - k_1, k_4 - k_2a_{12}, k_4 - k_3a_{13}).$$

We now present a classification scheme of the line-soliton solutions by identifying the asymptotic line-solitons as $y \rightarrow \pm\infty$. We denote a line-soliton solution by (N_-, N_+) -soliton whose asymptotic form consists of N_- line-solitons as $y \rightarrow -\infty$ and N_+ line-solitons for $y \rightarrow \infty$ in the xy -plane as shown in Figure 5.3. The following Proposition provides a general result characterizing the

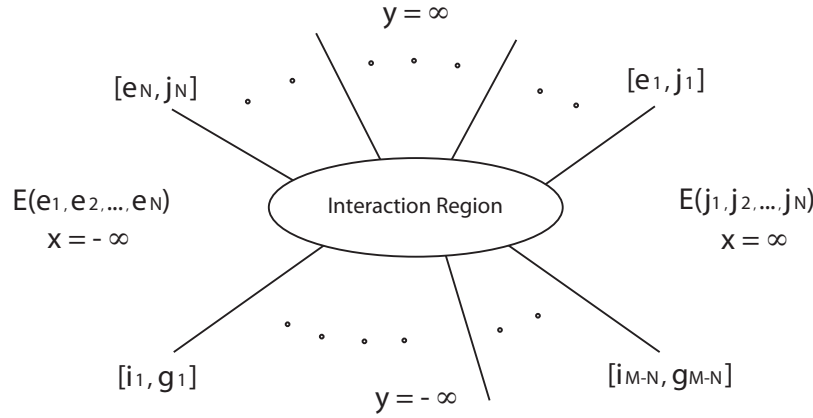


FIGURE 5.3. (N_-, N_+) -soliton solution. The asymptotic line-solitons are denoted by their index pairs $[e_n, j_n]$ and $[i_m, g_m]$. The sets $\{e_1, e_2, \dots, e_N\}$ and $\{g_1, g_2, \dots, g_{M-N}\}$ indicate pivot and non-pivot indices, respectively. Here $N_- = M - N$ and $N_+ = N$ for the τ -function on $\text{Gr}(N, M)$, and $E(\cdot, \dots, \cdot)$ represents the dominant exponential in that region.

asymptotic line-solitons of the (N_-, N_+) -soliton solutions.

Proposition 5.2. *Let $\{e_1, e_2, \dots, e_N\}$ be the pivot indices, and let $\{g_1, g_2, \dots, g_{M-N}\}$ be the non-pivot indices of an irreducible and totally non-negative $N \times M$ matrix A . Then the soliton solution generated by the τ -function in (5.1) with the matrix A has*

- (a) N asymptotic line-solitons as $y \rightarrow \infty$, each defined uniquely by the line $[e_n, j_n]$ for some j_n , and
- (b) $M - N$ asymptotic line-solitons as $y \rightarrow -\infty$, each defined uniquely by the line $[i_m, g_m]$ for some i_m .

Proof. Let $[i, j]$ denote an asymptotic line-soliton for $y \gg 0$, and let it correspond to the dominant exponentials $E(i, m_2, \dots, m_N)$ and $E(j, m_2, \dots, m_N)$ in (5.1). Hence in particular, the coefficient minor $\xi(i, m_2, \dots, m_N) \neq 0$. Assume that A_i is not a pivot column. Then A_i must be spanned by the pivot columns of A to its left since A is in RREF. That is, $A_i = \sum_{r=1}^n c_r A_{e_r}$, where $e_n < i$. Therefore,

$$\xi(i, m_2, \dots, m_N) = \sum_{r=1}^n c_r \xi(e_r, m_2, \dots, m_N) \neq 0.$$

Suppose c_s is the coefficient of the first nonzero term in the above sum, then the minor $\xi(e_s, m_2, \dots, m_N) \neq 0$, implying that the corresponding exponential $E(e_s, m_2, \dots, m_N)$ is present in the τ -function. Furthermore, from (5.5) $\eta_{e_s} > \eta_i$ along $[i, j]$, so that $E(e_s, m_2, \dots, m_N) > E(i, m_2, \dots, m_N)$. But this is impossible as $E(i, m_2, \dots, m_N)$ is the dominant exponential in the τ -function along $[i, j]$. Therefore, A_i must be a pivot column, proving part (a) of the Theorem.

Now suppose $[i, j]$ is an asymptotic line-soliton for $y \ll 0$ and that it corresponds to the dominant exponentials $E(i, m_2, \dots, m_N)$ and $E(j, m_2, \dots, m_N)$ in (5.1). Then the nonzero minor $\xi(i, m_2, \dots, m_N) \neq 0$ which implies that the column set $B = \{A_i, A_{m_2}, \dots, A_{m_N}\}$ forms a basis for \mathbb{R}^N . In particular, B spans the column A_j , i.e., $A_j = cA_i + \sum_{r=2}^N c_r A_{m_r}$. Then,

$$\xi(i, m_2, \dots, m_{r-1}, j, m_{r+1}, \dots, m_N) = c_r \xi(i, m_2, \dots, m_N), \quad r = 2, \dots, N.$$

If A_j is a pivot column, then there is at least one index s such that $m_s > j$. Then it follows from (5.5) that $\eta_{m_s} > \eta_j$ along the line $[i, j]$, and for $y \ll 0$, we have $E(i, m_2, \dots, m_{s-1}, j, m_{s+1}, \dots, m_N) \gg E(i, m_2, \dots, m_N)$. But this contradicts with $E(i, m_2, \dots, m_N)$ being dominant.

To prove uniqueness of each index pair, first consider $y > 0$, and assume that there are two line-solitons $[e_n, j_n]$ and $[e_n, j'_n]$ with a given pivot index e_n , and where $j_n > j'_n$. These line-solitons are localized along the lines $x + (k_{e_n} + k_{j_n})y = \text{constant}$ and $x + (k_{e_n} + k_{j'_n})y = \text{constant}$ with $k_{e_n} + k_{j_n} > k_{e_n} + k_{j'_n}$. Notice from Proposition 5.1 that the phase θ_{e_n} is replaced by the phase θ_{j_n} in the exponential term which is dominant immediately to the right of the $[e_n, j_n]$ -soliton. On the other hand, θ_{e_n} must be present in the exponential term which is dominant immediately to the left of the $[e_n, j'_n]$ -soliton. Consequently, there must be an intermediate line-soliton $[e_m, e_n]$ for some pivot index $e_m < e_n$, localized along the line $x + (k_{e_m} + k_{e_n})y = \text{constant}$, which must satisfy $k_{e_m} + k_{e_n} > k_{e_n} + k_{j'_n}$. But the latter inequality is impossible due to ordering of the indices $e_m < e_n < j'_n$ and the ordering $k_r < k_s$ for $r < s$, giving the required contradiction. Hence for a given pivot index e_n there is a unique line-soliton $[e_n, j_n]$ as $y \rightarrow \infty$. The uniqueness proof for the $[i_m, g_m]$ -solitons for $m = 1, \dots, M - N$, and as $y \rightarrow -\infty$, is similar. \square

The unique index pairings in Proposition 5.2 have a combinatorial interpretation. Let $[M] := \{1, 2, \dots, M\}$ be the integer set and recall that $\{e_1, \dots, e_N\} \cup \{g_1, \dots, g_{M-N}\}$ is a disjoint partition of $[M]$. Define the pairing map $\pi : [M] \rightarrow [M]$ according to parts (a) and (b) of Proposition 5.2 as follows:

$$(5.6) \quad \begin{cases} \pi(e_n) &= j_n, & n = 1, 2, \dots, N, \\ \pi(g_m) &= i_m, & m = 1, 2, \dots, M - N, \end{cases}$$

where e_n and g_m are respectively, the pivot and non-pivot indices of the A -matrix. Then as a consequence of Proposition 5.2 we have the following.

Corollary 5.1. *The map $\pi : [M] \rightarrow [M]$ is a bijection. That is, $\pi \in \mathcal{S}_M$, where \mathcal{S}_M is the group of permutations for the index set $[M]$.*

Proof. We prove this by finding an explicit relation among those parameters $\{k_{i_m}, k_{j_n}\}$ and $\{k_{e_n}, k_{g_m}\}$: For this purpose, we consider the integrals C_n in (3.15) for the $(M - N, N)$ -soliton solutions obtained in Proposition 5.2. Here we recall that

$$C_n = \int_{-\infty}^{\infty} v_{n+1}(x, y, \dots) dx, \quad n = 1, 2, \dots, M,$$

where $v_{n+1} = -\partial_x S_n(-\tilde{\partial}) \ln \tau$ (see (3.14)). Note that for $|y| \gg 0$ the τ -function is asymptotically equal to a single exponential function in each of the M asymptotic sectors in the xy -plane (see Figure 5.3). That is, in each asymptotic sector the τ -function has the form

$$\tau \approx \xi(m_1, \dots, m_N) E(m_1, \dots, m_N),$$

and all other terms are exponentially small in comparison with this dominant exponential. Then we have

$$\ln \tau \approx \sum_{j=1}^N \left(\sum_{n=1}^{\infty} k_{m_j}^n t_n \right) + c,$$

where c is a constant determined by $\xi(m_1, \dots, m_N)$ and the k -parameters. We now calculate C_n using Proposition 5.2 for $y \gg 0$ and $y \ll 0$.

For $y \gg 0$, there are $N + 1$ asymptotic regions, and the boundary of adjacent regions is given by the line $[e_n, j_n]$ for $n = 1, \dots, N$. Recall from Proposition 5.1 that the dominant exponential terms in adjacent regions differ by only one phase. We now integrate v_{n+1} along a horizontal line which passes through those asymptotic regions, by using the above expression for $\ln \tau$ and the fact that $S_n(-\tilde{\partial}) = -\frac{1}{n} \partial_{t_n} + (\text{h.o.d.})$. Since the contribution of the exponentially small terms vanish as $y \rightarrow \infty$, we then obtain

$$C_n = \frac{1}{n} \sum_{r=1}^N (k_{j_r}^n - k_{e_r}^n), \quad n = 1, 2, \dots,$$

where each term in the sum is the contribution to the integral from the pair of dominant exponentials in adjacent asymptotic regions for $y \gg 0$.

For $y \ll 0$, we follow the similar arguments as above and integrate v_{n+1} along a line across the $M - N + 1$ asymptotic regions separated by the lines $[i_m, g_m]$ for $m = 1, \dots, M - N$. We then obtain another expression for the integral C_n , namely

$$C_n = \frac{1}{n} \sum_{s=1}^{M-N} (k_{g_s}^n - k_{i_s}^n), \quad n = 1, 2, \dots$$

Since C_n do not depend on y , the two expressions obtained for $y \gg 0$ and $y \ll 0$ must be the same, that is, we obtain the relations,

$$C_n = \frac{1}{n} \sum_{r=1}^N (k_{j_r}^n - k_{e_r}^n) = \frac{1}{n} \sum_{s=1}^{M-N} (k_{g_s}^n - k_{i_s}^n).$$

Rearranging the terms, we have for $n = 1, 2, \dots, M$

$$\begin{aligned} \sum_{j=1}^M k_j^n &= \sum_{r=1}^N k_{e_r}^n + \sum_{s=1}^{M-N} k_{g_s}^n \\ &= \sum_{r=1}^N k_{j_r}^n + \sum_{s=1}^{M-N} k_{i_s}^n. \end{aligned}$$

Then Lemma 5.1 implies that there is a one-to-one correspondence between two sets $\{k_{e_r}, k_{g_s}\}$ and $\{k_{j_r}, k_{i_s}\}$. Furthermore, since $\{k_{e_r}, k_{g_s}\}$ is a set of distinct elements, then so is $\{k_{j_r}, k_{i_s}\}$. This proves that π is a bijection. \square

Note that the permutation π defined by (5.6) has no fixed point because $\pi(e_n) = j_n > e_n$, $n = 1, \dots, N$ and $\pi(g_m) = i_m < g_m$, $m = 1, \dots, M - N$. Such permutations are called *derangements*. Moreover, π have exactly N *excedances* defined as follows: an element $l \in [M]$ is an *excedance* of π if $\pi(l) > l$. The excedance set of π in (5.6) is the set of pivot indices $\{e_1, e_2, \dots, e_N\}$. We can now summarize the results of Proposition 5.2 and Corollary 5.1 as follows:

Theorem 5.4. *Let A be an $N \times M$ irreducible matrix which gives a coordinate of a non-negative cell of the Grassmannian $Gr(N, M)$. Then the tau-function (5.1) associated with the A -matrix generates an $(M - N, N)$ -soliton solutions. The total M asymptotic line-solitons associated with each of these solutions induce a pairing map π defined by (5.6). Moreover, π is a derangement of the index set $[M]$ with N excedances given by the pivot indices $\{e_1, e_2, \dots, e_N\}$ of the A -matrix in RREF.*

The derangements $\pi \in \mathcal{S}_M$ are represented by linear chord diagrams with the arrows above the line pointing from e_n to j_n for $n = 1, 2, \dots, N$, while arrows below the line point from g_m to i_m for $m = 1, 2, \dots, M - N$. Figure 5.4 illustrates the time evolution of a $(3, 3)$ -soliton solution. The chord diagram shows all asymptotic line-solitons for $y \rightarrow \pm\infty$.

6. (2, 2)-SOLITON SOLUTIONS

In this section we discuss all soliton solutions of the KP equation generated by the 2×4 irreducible A -matrices with nonnegative minors. Theorem 5.4 implies that each of the soliton solutions consists of two asymptotic line-solitons as $y \rightarrow \pm\infty$, that is, we have $(2, 2)$ -soliton solutions. We outline below the classification scheme for the $(2, 2)$ -soliton solutions. First we note that there are only two types of irreducible matrices for the size 2×4 , i.e.

$$\begin{pmatrix} 1 & 0 & -c & -d \\ 0 & 1 & a & b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & a & 0 & -c \\ 0 & 0 & 1 & b \end{pmatrix},$$

The condition for nonnegative minors implies that the constants a, b, c and d must be non-negative. In the first case, one can easily see that $ad = 0$ is impossible because then $\xi(3, 4) < 0$ from the irreducibility. Then there are 5 cases for nonnegative minors with $ad \neq 0$, which are (1) $ad - bc > 0$, (2) $ad - bc = 0$, (3) $b = 0, c \neq 0$, (4) $c = 0, b \neq 0$, and (5) $b = c = 0$. For the second case, since $ab \neq 0$ due to irreducibility, we have only two cases: (1) $c = 0$, (2) $c \neq 0$. Thus we have total seven

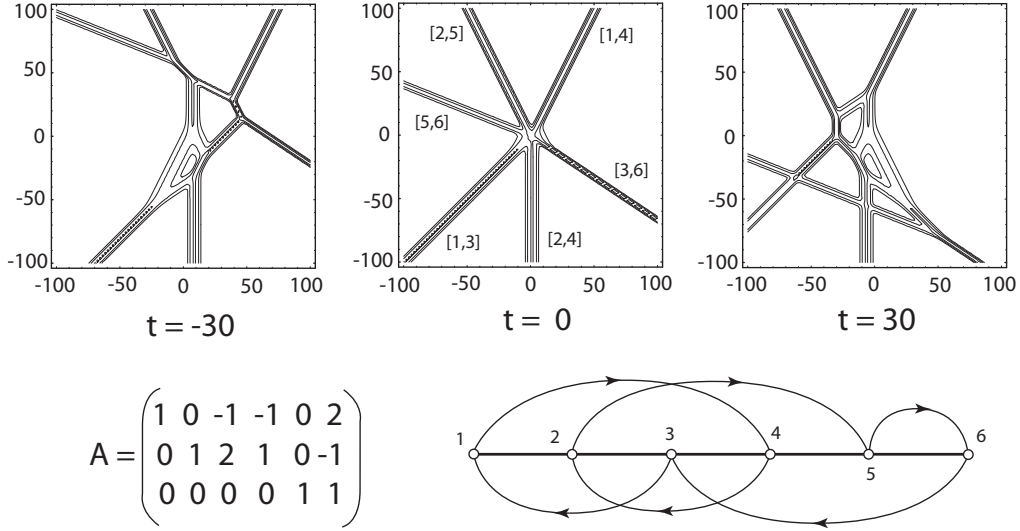


FIGURE 5.4. The time evolution of a $(3,3)$ -soliton solution. The permutation of this solution is $\pi = (451263)$. The k -parameters are chosen as $(k_1, k_2, \dots, k_6) = (-1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2})$. The dominant exponential for $x \ll 0$ is $E(1, 2, 5)$, and each dominant exponential is obtained through the derangement representing the solution, e.g. after crossing $[5, 6]$ -soliton in the clockwise direction, the dominant exponential becomes $E(1, 2, 6)$. That is, $[5, 6]$ -soliton is given by the balance of those two exponentials.

different types of A -matrices, and each A -matrix gives a different $(2, 2)$ -soliton solution. We now summarize the results for all seven cases with 2×4 irreducible and totally non-negative A -matrices:

- (a) $\pi = (3412)$: This case gives *T-type* 2-soliton solution, and the asymptotic line-solitons are $[1, 3]$ - and $[2, 4]$ -types for $|y| \rightarrow \infty$. This type of solution was first obtained as the solution of the *Toda* lattice hierarchy [4], and this is why we call it “T-type” (see also [17]). The A -matrix is given by

$$A = \begin{pmatrix} 1 & 0 & -c & -d \\ 0 & 1 & a & b \end{pmatrix},$$

where $a, b, c, d > 0$ are free parameters with $ad - bc > 0$. This is the generic solution on the maximum dimensional cell (four free parameters) of $\text{Gr}(2, 4)$, and contains all possible line-solitons in the process of interaction, i.e. $[i, j]$ -solitons for any $\{i, j\}$ pairs (see the next section for details).

- (b) $\pi = (4312)$: The asymptotic line-solitons are given by

- (i) $[1, 4]$ - and $[2, 3]$ -solitons in $y \gg 0$
- (ii) $[1, 3]$ - and $[2, 4]$ -solitons in $y \ll 0$.

The A -matrix is given by

$$A = \begin{pmatrix} 1 & 0 & -b & -c \\ 0 & 1 & a & 0 \end{pmatrix},$$

where $a, b, c > 0$ are free parameters. Note that two line-solitons for $y \ll 0$ are the same types as in the T-type solitons. We also observe all possible line-solitons including intermediate ones, but not for all t unlike the T-type.

- (c) $\pi = (3421)$: The asymptotic line-solitons are given by

- (i) $[1, 3]$ - and $[2, 4]$ -solitons in $y \gg 0$
- (ii) $[1, 4]$ - and $[2, 3]$ -solitons in $y \ll 0$.

The A -matrix is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & -c \\ 0 & 1 & a & b \end{pmatrix},$$

where $a, b, c > 0$ are positive free parameters. This solution can be considered as a dual of the previous case (b), that is, two sets of line-solitons for $y \gg 0$ and $y \ll 0$ are exchanged. The example discussed in Example 5.2 corresponds to this solution (see Figure 5.2).

- (d) $\pi = (2413)$: The asymptotic line-solitons are given by

- (i) $[1, 2]$ - and $[2, 4]$ -solitons in $y \gg 0$
- (ii) $[1, 3]$ - and $[3, 4]$ -solitons in $y \ll 0$.

The A -matrix is given by

$$A = \begin{pmatrix} 1 & 0 & -c & -d \\ 0 & 1 & a & b \end{pmatrix},$$

where $a, b, c, d > 0$ with $\xi(3, 4) = ad - bc = 0$.

- (e) $\pi = (3142)$: The asymptotic line-solitons are given by

- (i) $[1, 3]$ - and $[3, 4]$ -solitons in $y \gg 0$
- (ii) $[1, 2]$ - and $[2, 4]$ -solitons in $y \ll 0$.

The A -matrix is given by

$$A = \begin{pmatrix} 1 & a & 0 & -c \\ 0 & 0 & 1 & b \end{pmatrix},$$

where $a, b, c > 0$. This solution is dual to the previous one (d) in the sense that the missing minors are switched by $\xi(3, 4) \leftrightarrow \xi(1, 2)$. This case will be further discussed in the next subsection to describe a connection with the Mach reflection in shallow water waves.

- (f) $\pi = (4321)$: This case gives *P-type* 2-soliton solution, and the asymptotic line-solitons are $[1, 4]$ - and $[2, 3]$ -types for $|y| \rightarrow \infty$. This type of solutions fits better with the *physical* assumption for the derivation of the KP equation, i.e. a quasi-two dimensionality with weak y -dependence. This is why we call it “P-type” (see [17]). The situation is similar to the KdV case, for example, two solitons must have different amplitudes, $A_{[1,4]} > A_{[2,3]}$. The A -matrix is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & -b \\ 0 & 1 & a & 0 \end{pmatrix}.$$

- (g) $\pi = (2143)$: This case gives *O-type* 2-soliton solution, and the asymptotic line-solitons are $[1, 2]$ - and $[3, 4]$ -types for $|y| \rightarrow \infty$. The letter “O” for this type is due to the fact that this solution was *originally* found to describe the two-soliton solution, see for example [11], (see also [17]). Interaction properties for solitons with equal amplitude has been discussed using this solution. However this solution becomes singular, when those solitons are almost parallel to the y -axis; in this case, the order of the k -parameters is reversed, that is, $k_2 > k_3$ leading to $E(2, 3) < 0$). This is contrary to the assumption of the quasi-two dimensionality for the KP equation [22]. We will discuss this issue in the next section. The A -matrix is given by

$$A = \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 0 & 1 & b \end{pmatrix}.$$

In Figure 6.1, we list the chord diagrams for all those seven cases. One should note that any derangement of \mathcal{S}_4 with exactly two excedances should be one of the graphs. This uniqueness in the general case has been used to count the number of totally non-negative Grassmann cells [30, 35]. We also mention that these seven types of soliton solutions have been found by applying recursive binary Darboux transformations (see Appendix in [5]).

Now let us describe the details of some of the $(2, 2)$ -soliton solutions, which will be important for an application of those solutions to shallow water problem discussed in the next Section. In particular, we explain how the A -matrix uniquely determines the structure of the corresponding soliton solution such as the location of the solitons and their phase shifts.

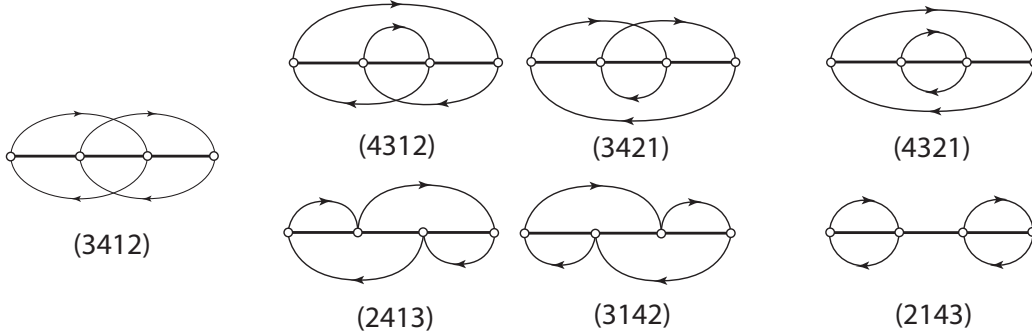


FIGURE 6.1. The chord diagrams for seven different types of $(2, 2)$ -soliton solutions. Each diagram corresponds to a totally non-negative Grassmannian cell in $\text{Gr}(2, 4)$.

6.1. O-type soliton solutions. This is the original two-soliton solution, and the solutions correspond to the chord diagram of $\pi = (2143)$. A solution of this type consists of two full line-solitons of $[1, 2]$ and $[3, 4]$ (see Figure 6.2). Note here that they have phase shifts due to their collision. Let us describe explicitly the structure of the solution of this type: The τ -function defined in (5.1) for this case is given by

$$\tau = E(1, 3) + bE(1, 4) + aE(2, 3) + abE(2, 4),$$

where $a, b > 0$ are the free parameters given in the A -matrix listed in the previous section. As we will show that those two parameters can be used to fix the locations of those solitons, that is, they are determined by the asymptotic data of the solution for large $|y|$.

For the later application of the solution, we assume that $[1, 2]$ -soliton has a “positive” y -component in the wave-vector (i.e. $\tan \Psi_{[1,2]} < 0$), and $[3, 4]$ -soliton has a “negative” y -component, (i.e. $\tan \Psi_{[3,4]} > 0$, see Figure [?]). (Recall that any line-soliton has a “negative” x -component in the wave-vector.) Then for the region with large positive x , we have $[1, 2]$ -soliton in $y > 0$ and $[3, 4]$ -soliton in $y < 0$. Those solitons are obtained by the balance between two exponential terms in the τ -function:

For $[1, 2]$ -soliton in $x > 0$ (and $y \gg 0$), we have the dominant balance between $E(1, 4)$ and $E(2, 4)$. Then the τ -function can be written in the following form,

$$\begin{aligned} \tau &\approx bE(1, 4) + abE(2, 4) \\ &= 2be^{\theta_4 + \frac{1}{2}(\theta_1 + \theta_2)} \cosh \frac{1}{2}(\theta_1 - \theta_2 + \theta_{12}^\pm), \end{aligned}$$

which leads to the $[1, 2]$ -soliton solution in the region near $\theta_1 \approx \theta_2$ for large $x > 0$,

$$u = 2\partial_x^2 \ln \tau \approx \frac{1}{2}(k_2 - k_1)^2 \text{sech}^2 \frac{1}{2}(\theta_1 - \theta_2 + \theta_{12}^+).$$

Here the phase shift θ_{12}^+ (+ indicates $x > 0$) is given by

$$(6.1) \quad \theta_{12}^+ = \ln \frac{k_4 - k_1}{k_4 - k_2} - \ln a \quad \text{i.e.} \quad a = \frac{k_4 - k_1}{k_4 - k_2} e^{-\theta_{12}^+}.$$

The parameter a determines the location of the $[1, 2]$ -soliton. For example, one can choose a so that $\theta_{12}^+ = 0$, which implies that the $[1, 2]$ -soliton is passing through the origin at $t = 0$.

For $[3, 4]$ -soliton in $x > 0$ (and $y \ll 0$), from the balance $\tau \approx aE(2, 3) + abE(2, 4)$, we have

$$u \approx \frac{1}{2}(k_4 - k_3)^2 \text{sech}^2 \frac{1}{2}(\theta_3 - \theta_4 + \theta_{34}^+),$$

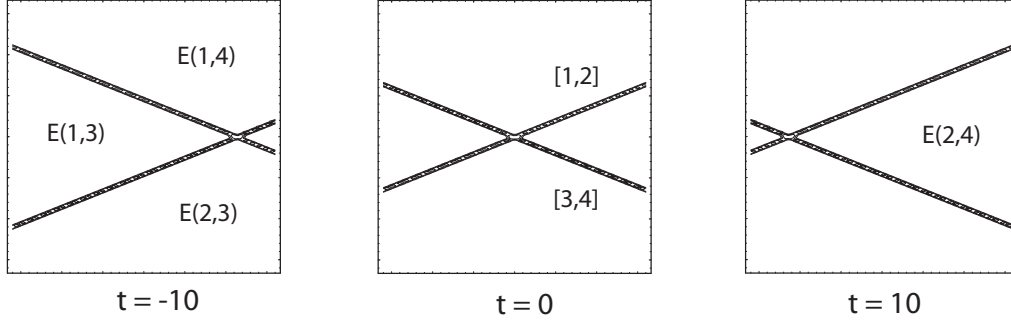


FIGURE 6.2. The time evolution of an O-type soliton solution. Each $E(i, j)$ indicates the dominant exponential in that region. The parameters are chosen as $(k_1, k_2, k_3, k_4) = (-\frac{9}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{9}{4})$, so that both line-solitons have the same amplitude, $A_{[1,2]} = A_{[3,4]} = 2$, and the directions of the wave-vectors are given by $\Psi_{[3,4]} = -\Psi_{[1,2]} = \tan^{-1} \frac{5}{2} \approx 68.2^\circ$.

with the phase shift,

$$(6.2) \quad \theta_{34}^+ = \ln \frac{k_3 - k_2}{k_4 - k_2} - \ln b \quad \text{i.e.} \quad b = \frac{k_3 - k_2}{k_4 - k_2} e^{-\theta_{34}^+}.$$

Thus the parameter b determines the location of $[3, 4]$ -soliton, and one can choose appropriate b so that $\theta_{34}^+ = 0$. Thus the parameters in the A -matrix can be determined by the asymptotic data of the phase shifts θ_{12}^+ and θ_{34}^+ for $x \gg 0$ and $|y| \gg 0$.

Now calculating the phase shift for the $[3, 4]$ -soliton in $x < 0$, one can find the total phase shift along the line-soliton: From the balance between the terms with $E(1, 3)$ and $bE(1, 4)$, and following the previous arguments, we obtain

$$\theta_{34}^- = \ln \frac{k_3 - k_1}{k_4 - k_1} - \ln b.$$

It is interesting to note that the sum of the phase shifts for those solitons are conserved in the y -direction for each A -matrix,

$$(6.3) \quad \theta_{12}^+ + \theta_{34}^- = \theta_{34}^+ + \theta_{12}^- = \ln \frac{k_3 - k_1}{k_4 - k_2} - \ln(ab).$$

We measure the total phase shift for each soliton in the reference of the soliton in $x \ll 0$, that is, the total phase shift is defined by $\theta_{34} = \theta_{34}^+ - \theta_{34}^-$ (recall that the phase shift is generated by interacting with other soliton, and the effect of the interaction propagates only in the “positive” x -direction). We then obtain the well-known formula for the phase shift θ_{34} (see for example [14]),

$$\theta_{34} = \ln \frac{(k_4 - k_1)(k_3 - k_2)}{(k_3 - k_1)(k_4 - k_2)}.$$

Notice that this does not depend on the A -matrix. One can also see from (6.3) that the total phase shift of $[1, 2]$ -soliton is the same as that of the $[3, 4]$ -soliton, i.e.

$$\theta_{12} = \theta_{12}^+ - \theta_{12}^- = \theta_{34} = \ln \Delta_O,$$

where we note

$$\Delta_O := \frac{(k_3 - k_2)(k_4 - k_1)}{(k_4 - k_2)(k_3 - k_1)} = 1 - \frac{(k_2 - k_1)(k_4 - k_3)}{(k_4 - k_2)(k_3 - k_1)} < 1.$$

This implies that $\theta_{12} = \theta_{34} < 0$, and each $[i, j]$ -soliton shifts in x with

$$\Delta x_{[i,j]} = \frac{1}{k_j - k_i} \theta_{ij}.$$

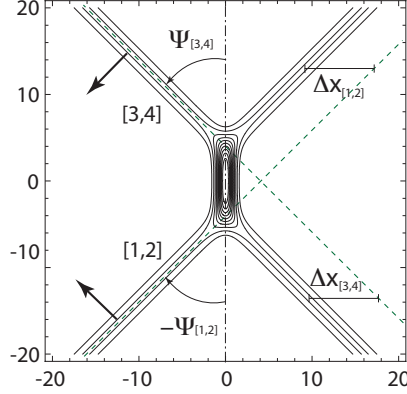


FIGURE 6.3. O-type interaction for two equal amplitude solitons. The parameters k_i 's are taken by $(k_1, k_2, k_3, k_4) = (-1 - 10^{-4}, -10^{-4}, 10^{-4}, 1 + 10^{-4})$, which give $A_0 = A_{[1,2]} = A_{[3,4]} = \frac{1}{2}$ and $\Psi_0 = \tan \Psi_{[3,4]} = -\tan \Psi_{[1,2]} = 1 + 2 \times 10^{-4}$ (i.e. $\Psi_{[3,4]} \approx 45.0057$). The constants a, b in the A -matrix are chosen so that the center of interaction point is located at the origin (see (6.4)), and $u_{\max} = u(0, 0, 0) \approx 1.96$ (about four times larger than each soliton amplitude A_0).

The negative phase shifts $\Delta x_{[1,2]} < 0$ and $\Delta x_{[3,4]} < 0$ indicate an *attractive* force in the interaction, and if the amplitudes are the same, i.e. $k_2 - k_1 = k_4 - k_3$, then their phase shifts are the same, i.e. $\Delta x_{[1,2]} = \Delta x_{[3,4]} = \frac{1}{k_4 - k_3} \theta_{34} < 0$. Figure 6.3 illustrates an O-type interaction of two solitons which have the same amplitude, $A_{[1,2]} = A_{[3,4]} = A_0$, and are symmetric with respect to the y -axis, $\Psi_{[3,4]} = -\Psi_{[1,2]} = \Psi_0$.

O-type soliton solution has a steady X-shape with phase shifts in both line-solitons. One can also find the formula of the maximum amplitude which occurs at the center of intersection point (center of the X-shape): We place the soliton solution so that the origin $(0, 0)$ is the center of the X-shape at $t = 0$. This implies that the sum of the phase shifts vanishes for each soliton, i.e.

$$(6.4) \quad \begin{cases} \theta_{12}^+ + \theta_{12}^- = \ln \frac{(k_4 - k_1)(k_3 - k_1)}{(k_4 - k_2)(k_3 - k_2)} - 2 \ln a = 0, \\ \theta_{34}^+ + \theta_{34}^- = \ln \frac{(k_3 - k_1)(k_3 - k_2)}{(k_4 - k_1)(k_4 - k_2)} - 2 \ln b = 0. \end{cases}$$

Those determine a and b in the A -matrix. Then the τ -function becomes

$$\begin{aligned} \tau &= (k_3 - k_1) \left(E_1 E_3 + E_2 E_4 + \sqrt{\Delta_0} (E_1 E_4 + E_2 E_3) \right) \\ &\equiv E_1 \left(E_3 + \sqrt{\Delta_0} E_4 \right) + E_2 \left(\sqrt{\Delta_0} E_3 + E_4 \right) \\ &\equiv E_{12}^+ \left(E_{34}^+ + \sqrt{\Delta_0} E_{34}^- \right) + E_{12}^- \left(\sqrt{\Delta_0} E_{34}^+ + E_{34}^- \right) \\ &\equiv \cosh \Theta^+ + \sqrt{\Delta_0} \cosh \Theta^-, \end{aligned}$$

where $\alpha \equiv \beta$ implies that α and β give the same solution u , and E_{ij}^\pm and Θ^\pm are defined by

$$E_{ij}^\pm := \exp \left(\pm \frac{\theta_i - \theta_j}{2} \right), \quad \Theta^\pm := \frac{1}{2} [(\theta_1 - \theta_2) \pm (\theta_3 - \theta_4)].$$

It is then not difficult to see that $u(x, y, 0)$ attains the maximum at the origin, and we get the maximum value $u_{\max} := u(0, 0, 0)$, which is

$$\begin{aligned}
 u_{\max} &= \frac{1}{2} ((k_1 - k_2)^2 + (k_3 - k_4)^2) + \frac{1 - \sqrt{\Delta_O}}{1 + \sqrt{\Delta_O}} (k_1 - k_2)(k_3 - k_4) \\
 (6.5) \quad &= A_{[1,2]} + A_{[3,4]} + 2 \frac{1 - \sqrt{\Delta_O}}{1 + \sqrt{\Delta_O}} \sqrt{A_{[1,2]} A_{[3,4]}}.
 \end{aligned}$$

(See also [9, 32].) Since $0 < \Delta_O < 1$, we have the bound

$$A_{[1,2]} + A_{[3,4]} < u_{\max} < \left(\sqrt{A_{[1,2]}} + \sqrt{A_{[3,4]}} \right)^2.$$

It is also interesting to note that the formula Δ_O has critical cases at the values $k_1 = k_2$ or $k_2 = k_3$ or $k_3 = k_4$. For the case with $k_1 = k_2$ or $k_3 = k_4$ (i.e. $\Delta_O = 1$), one can see that one of the line-soliton becomes small, and the limit consists of just one-soliton solution. On the other hand, for the case $k_2 = k_3$ (i.e. $\Delta_O = 0$), the τ -function has only three terms, which corresponds to a solution showing a Y-shape interaction (i.e. the phase shift becomes infinity and the middle portion of the interaction stretches to infinity). This limit has been discussed in [22, 26] as a resonant interaction of three waves to make Y-shape soliton. This limit gives a critical angle between those solitons which can be found as follows: First let us express each k_j parameter in terms of the amplitude and the slope,

$$\begin{aligned}
 k_{1,2} &= \frac{1}{2} \left(\tan \Psi_{[1,2]} \mp \sqrt{2A_{[1,2]}} \right), \\
 k_{3,4} &= \frac{1}{2} \left(\tan \Psi_{[3,4]} \mp \sqrt{2A_{[3,4]}} \right),
 \end{aligned}$$

where the angle $\Psi_{[i,j]}$ is measured in the counterclockwise direction from the y -axis (see Figure 6.3). In particular, we have

$$\tan \Psi_{[1,2]} = -\sqrt{2A_{[1,2]}} + 2k_2, \quad \tan \Psi_{[3,4]} = \sqrt{2A_{[3,4]}} + 2k_3.$$

Without loss of generality, let us consider the special case when both solitons are of equal amplitude and symmetric with respect the y -axis i.e., $A_{[1,2]} = A_{[3,4]} = A_0$ and $\Psi_{[3,4]} = -\Psi_{[1,2]} = \Psi_0 > 0$. This corresponds to setting $k_1 = -k_4$ and $k_2 = -k_3$. Then, for fixed amplitude A_0 , the angle Ψ_0 has a lower bound given by

$$\tan \Psi_0 = \sqrt{2A_0} + 2k_3 \geq \sqrt{2A_0} := \tan \Psi_c.$$

The lower bound is achieved in the limit $k_2 = k_3 = 0$, and the critical angle Ψ_c is given by

$$(6.6) \quad \Psi_c = \tan^{-1} \sqrt{2A_0}.$$

Note that this function is monotone increasing in A_0 , and for $A_0 = \frac{1}{2}$ the critical angle occurs when the two solitons intersect perpendicularly. In this symmetric case, from (6.5), the maximum amplitude given at the center of interaction can be calculated as

$$(6.7) \quad u_{\max} = \frac{4A_0}{1 + \sqrt{\Delta_O}}, \quad \text{with} \quad \Delta_O = 1 - \frac{2A_0}{\tan^2 \Psi_0}.$$

Thus, at the critical angle $\Psi_0 = \Psi_c$ (i.e., $\Delta_O = 0$), we have $u_{\max} = 4A_0$ (see also [22, 9, 32]).

We also note that there are three other soliton solutions of (2, 2)-type which have the same limit. These are the cases marked by $\pi = (3412)$, (2413) and (3142), and their chord diagrams degenerate to the same graph of figure eight in the limit. We will discuss these cases in the following sections. Since (2413)- and (3142)-types are dual each other, we only discuss the (3142)-type.

One should note that if we use the form of the O-type solution even beyond the critical angle, i.e. $k_3 < k_2$, then the solution becomes singular (note that the sign of $E(2, 3)$ changes). In earlier works, this was considered to be an obstacle for using the KP equation to describe an interaction of two line-solitons with a smaller angle. On the contrary, the KP equation should give a better approximation

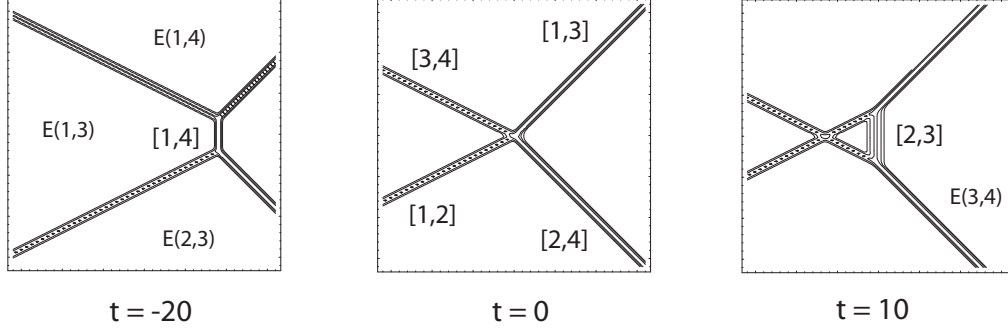


FIGURE 6.4. The time evolution of a (3142)-type soliton solution. The parameters are chosen as $(k_1, k_2, k_3, k_4) = (-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2})$. The amplitudes of those solitons are $A_{[1,3]} = A_{[2,4]} = 2$ and $A_{[1,2]} = A_{[3,4]} = \frac{1}{2}$. The directions of the wave-vectors are $\Psi_{[2,4]} = -\Psi_{[1,3]} = 45^\circ$. The critical angle is given by $\Psi_c = \tan^{-1} \sqrt{2A_0} \approx 63.4^\circ$, which also gives $\Psi_c = \Psi_{[3,4]} = -\Psi_{[1,2]}$.

to describe oblique interactions of solitons with smaller angles. Thus one should expect to have explicit solutions of the KP equation describing such phenomena. It turns out that the new types of (2, 2)-soliton solutions discussed above can indeed serve as good models for describing line-soliton interactions of solitons with small angles. We will show in the next section how these solutions are related to the *Mach reflections* discussed in [22].

6.2. (3142)-type soliton solutions. We consider a solution of this type which consists of two line-solitons for large positive x and two other line-solitons for large negative x . We then assume that the slopes of two solitons in each region have opposite signs, i.e. one in $y > 0$ and other in $y < 0$ (see Figure 6.4). The line-solitons for the (3142)-type solution can be determined from the balance between two appropriate exponential terms in its τ -function which has the form,

$$\tau = E(1, 3) + bE(1, 4) + aE(2, 3) + abE(2, 4) + cE(3, 4).$$

The solution contains three free parameters a, b and c , which can be used to determine the locations of three (out of four) asymptotic line-solitons (e.g. two in $x \gg 0$ and one in $x \ll 0$). Thus, the parameters are completely determined from the asymptotic data on large $|y|$.

Let us first consider the line-solitons in $x \gg 0$: There are two line-solitons which are $[1, 3]$ -soliton in $y \gg 0$ and $[2, 4]$ -soliton in $y \ll 0$. The $[1, 3]$ -soliton is obtained by the balance between the exponential terms $bE(1, 4)$ and $cE(3, 4)$, and the $[2, 4]$ -soliton is by the balance between $aE(2, 3)$ and $cE(3, 4)$. Consequently, the phase shifts of $[1, 3]$ - and $[2, 4]$ -solitons for $x \gg 0$ are given by

$$(6.8) \quad \theta_{13}^+ = \ln \frac{k_4 - k_1}{k_4 - k_3} + \ln \frac{b}{c}, \quad \theta_{24}^+ = \ln \frac{k_3 - k_2}{k_4 - k_3} + \ln \frac{a}{c}.$$

Choosing appropriate values for the ratios a/c and b/c , one can set $\theta_{13}^+ = \theta_{24}^+ = 0$, thus fixing the locations of those line-solitons as passing through the origin at $t = 0$. Notice that there is one free parameter left after this.

Now we consider the line-solitons in $x \ll 0$: They are $[3, 4]$ -soliton in $y \gg 0$ and $[1, 2]$ -soliton in $y \ll 0$. The phase shifts are given respectively by

$$\theta_{34}^- = \ln \frac{k_3 - k_1}{k_4 - k_1} - \ln b, \quad \theta_{12}^- = \ln \frac{k_3 - k_1}{k_3 - k_2} - \ln a$$

The four phase shifts and the three free parameters a, b, c are related by

$$a = \frac{k_3 - k_1}{k_3 - k_2} e^{-\theta_{12}^-}, \quad b = \frac{k_3 - k_1}{k_4 - k_1} e^{-\theta_{34}^-}, \quad c = \frac{k_3 - k_1}{k_4 - k_3} e^{-\theta_{13}^+ - \theta_{34}^-},$$

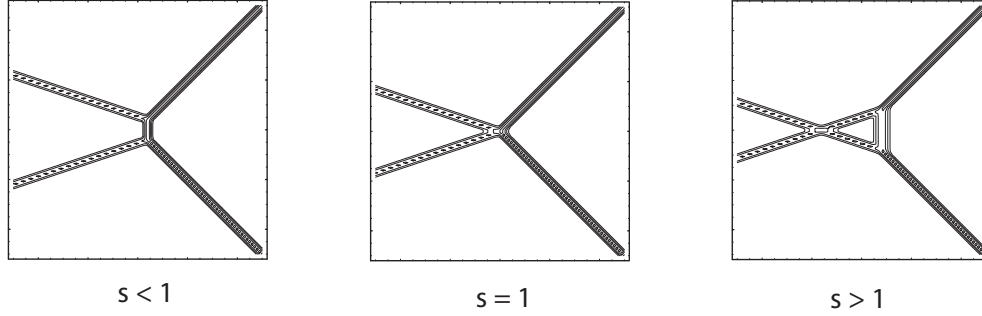


FIGURE 6.5. A (3142)-type soliton solution with the s -parameter. The k -parameters are taken as $(k_1, k_2, k_3, k_4) = (-2, -1, 1, 2)$. The parameters in the A -matrix are chosen as $a = \frac{3}{2}s, b = \frac{3}{4}s$ and $c = 3s$, so that $[1, 3]$ - and $[2, 4]$ -solitons meet at the origin, i.e. $\theta_{13}^+ = \theta_{24}^+ = 0$ (see (6.10)). Then at $s = 1$, all the solitons meet at the origin, i.e. the s -parameter shifts $[1, 2]$ - and $[3, 4]$ -solitons.

with the conservation of total phase shifts for $y \rightarrow \pm\infty$ (as in the case of O-type, see (6.3)),

$$\theta_{13}^+ + \theta_{34}^- = \theta_{24}^+ + \theta_{12}^-.$$

We then define the parameter representing this conservation,

$$(6.9) \quad s := \exp(-\theta_{13}^+ - \theta_{34}^-),$$

which leads to

$$(6.10) \quad a = \frac{k_3 - k_1}{k_3 - k_2} s e^{\theta_{24}^+}, \quad b = \frac{k_3 - k_1}{k_4 - k_1} s e^{\theta_{13}^+}, \quad c = \frac{k_3 - k_1}{k_4 - k_3} s.$$

The s -parameter represents the relative locations of the intersection point of the $[1, 3]$ - and $[3, 4]$ -solitons with the x -axis, in particular, $\theta_{13}^+ + \theta_{34}^- = 0$ when $s = 1$. Thus the parameters a, b and c are related to the locations of $[1, 3]$ -soliton (with θ_{13}^+), of $[2, 4]$ -soliton (with θ_{24}^+), and the intersection point of $[1, 3]$ - and $[3, 4]$ -solitons (with s).

Now we consider an example, in which the $[1, 3]$ - and $[2, 4]$ -solitons have the same amplitude ($A_{[1,3]} = A_{[2,4]} = A_0$) and they are symmetric with respect to the x -axis ($\Psi_{[2,4]} = -\Psi_{[1,3]} = \Psi_0$). Then in terms of the k -parameters, we have

$$k_3 - k_1 = k_4 - k_2 = \sqrt{2A_0}.$$

Also the symmetry of the wave-vectors, i.e. $\Psi_{[2,4]} = \Psi_0 = -\Psi_{[1,3]}$, gives

$$k_2 + k_4 = -(k_1 + k_3) = \tan \Psi_0.$$

This implies that we have

$$(6.11) \quad k_4 = -k_1 > 0, \quad k_3 = -k_2 > 0.$$

The angle Ψ_0 takes the value in $(0, \Psi_c)$, where the critical angle is given by the condition $k_2 = k_3 = 0$, i.e.

$$\Psi_c = \tan^{-1} \sqrt{2A_0} < \frac{\pi}{2}.$$

Notice that this formula is the same as that of the O-type soliton solution (see (6.6)). In this sense, these two cases have the same limit but from opposite directions namely, $\Psi_0 \rightarrow \Psi_c$ from above for the O-type, while $\Psi_0 \rightarrow \Psi_c$ from below for the (3142)-type. This fact will be important for the initial value problem discussed in Section 7.

From (6.11), one can easily deduce the following facts for $[1, 2]$ - and $[3, 4]$ -solitons in $x < 0$:

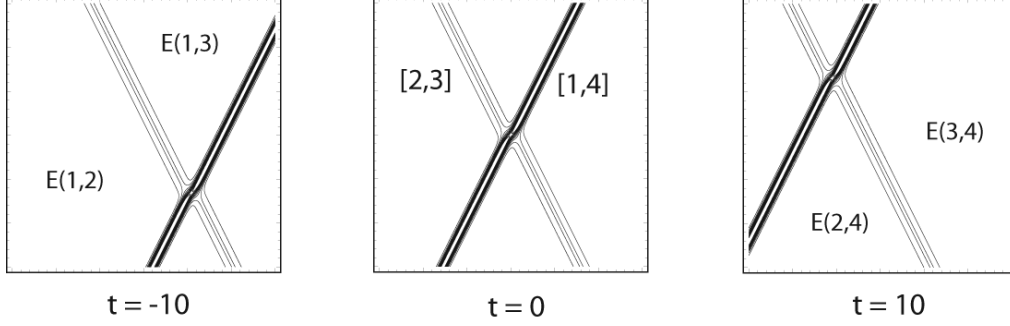


FIGURE 6.6. The time evolution of a P-type soliton solution. The parameters are chosen as $(k_1, k_2, k_3, k_4) = (-\frac{3}{2}, -\frac{1}{4}, \frac{3}{4}, 1)$. The amplitudes are $A_{[1,4]} = \frac{25}{8}$, $A_{[2,3]} = \frac{1}{2}$. The directions of the wave-vectors are given by $\Psi_{[2,3]} = -\Psi_{[1,4]} = \tan^{-1} \frac{1}{2} \approx 26.57^\circ$. The parameters in the A -matrix are chosen as $a = \frac{5}{3}$ and $b = 3$, so that the center of the interaction point is located at the origin at $t = 0$.

- (a) Those solitons have the same amplitude, i.e.

$$A_{[1,2]} = A_{[3,4]} = \frac{1}{2}(k_4 - k_3)^2 = \frac{1}{2}(k_4 + k_2)^2 = \frac{1}{2} \tan^2 \Psi_0.$$

Thus, if the $[1, 3]$ - and $[2, 4]$ -solitons in $x > 0$ are close to the y -axis (i.e. a small Ψ_0), then the amplitudes of the solitons in $x < 0$ are small; whereas at the critical angle $\Psi_0 = \Psi_c$, the solitons $[1, 2]$ and $[3, 4]$ in $x < 0$ take the maximum amplitude $A_{[1,2]} = A_{[3,4]} = A_0$.

- (b) The directions of the wave-vectors for the $[1, 2]$ and $[3, 4]$ -solitons are also symmetric, i.e.

$$\tan \Psi_{[3,4]} = -\tan \Psi_{[1,2]} = k_3 + k_4.$$

Moreover, the symmetry (6.11) implies that $\tan \Psi_{[3,4]} = k_4 - k_2 = \sqrt{2A_{[2,4]}} = \sqrt{2A_0}$, so

$$\Psi_{[3,4]} = \Psi_c = \tan^{-1} \sqrt{2A_0}.$$

Thus the directions of the wave-vectors for the $[1, 2]$ and $[3, 4]$ -solitons in $x < 0$ depend only on the amplitude of the solitons in $x > 0$ but not on their directions (i.e., angle of their V-shape).

Let us choose the parameters in the A -matrix for the (3142)-soliton solution appropriately, so that at $t = 0$ all the solitons intersect at the origin (see Figure 6.4). Then for $t > 0$, the resonant interaction between $[1, 3]$ - and $[3, 4]$ -solitons (as well as $[2, 4]$ - and $[1, 2]$ -solitons) generates an intermediate line-soliton (called “stem” soliton) which is $[1, 4]$ soliton. The amplitude of this soliton is given by

$$(6.12) \quad A_{[1,4]} = \frac{1}{2}(k_4 - k_1)^2 = 2k_4^2 = \frac{1}{2} \left(\sqrt{2A_0} + \tan \Psi_0 \right)^2.$$

Note here that at the critical angle $\Psi_0 = \Psi_c$, the amplitude takes the maximum $A_{[1,4]} = 4A_0$ (see [33, 29]).

For $t < 0$, the resonant interaction between $[1, 3]$ - and $[1, 2]$ -solitons (as well as $[2, 4]$ - and $[3, 4]$ -solitons) generates an intermediate line-soliton of $[2, 3]$ -soliton. The amplitude of $[2, 3]$ -soliton is given by

$$A_{[2,3]} = \frac{1}{2}(k_3 - k_2)^2 = \frac{1}{2} \left(\sqrt{2A_0} - \tan \Psi_0 \right)^2.$$

Because of the symmetry (6.11), both $[1, 4]$ - and $[2, 3]$ -solitons are parallel to the y -axis, i.e. $\tan \Psi_{[1,4]} = \tan \Psi_{[2,3]} = 0$.

6.3. P-type soliton solutions. This type of solution consists of two line-solitons, $[1, 4]$ and $[2, 3]$. We again assume that $[1, 4]$ -soliton has positive slope and $[2, 3]$ -soliton has negative slope (see Figure 6.6).

The phase shifts of $[1, 4]$ - and $[2, 3]$ -soliton in $x \gg 0$ are given by

$$\theta_{14}^+ = \ln \frac{k_3 - k_1}{k_4 - k_3} - \ln b, \quad \theta_{23}^+ = \ln \frac{k_4 - k_2}{k_4 - k_3} - \ln a.$$

Those parameters a and b in the A -matrix can be chosen, so that at $t = 0$ those solitons meet at the origin.

The phase shifts of those solitons in $x \ll 0$ are

$$\theta_{14}^- = \ln \frac{k_2 - k_1}{k_4 - k_2} - \ln b, \quad \theta_{23}^- = \ln \frac{k_2 - k_1}{k_3 - k_1} - \ln a.$$

The total phase shift of $[1, 4]$ -soliton is given by

$$\theta_{14} = \theta_{14}^+ - \theta_{14}^- = \ln \frac{(k_4 - k_2)(k_3 - k_1)}{(k_2 - k_1)(k_4 - k_3)} > 0.$$

Again this value is the same as the shift for $[2, 3]$ -soliton, i.e. $\theta_{23} = \theta_{23}^+ - \theta_{23}^- = \theta_{14}$. Because of the order $k_1 < k_2 < k_3 < k_4$, the phase shift is always positive, i.e.

$$(6.13) \quad \Delta x_{[1,4]} = \frac{1}{k_4 - k_1} \theta_{14} > 0.$$

This situation is similar to the case of the KdV solitons. Notice in particular that the total phase shift has the opposite sign to that of the O-type soliton solution [3], which indicates a *repulsive* force in the interaction.

As in the case of O-type, one can find the amplitude at the center of interaction: We calculate the sums of θ_{14}^\pm and θ_{23}^\pm , and set

$$\begin{cases} \theta_{14}^+ + \theta_{14}^- = \ln \frac{(k_2 - k_1)(k_3 - k_1)}{(k_4 - k_2)(k_4 - k_3)} - 2 \ln b = 0, \\ \theta_{23}^+ + \theta_{23}^- = \ln \frac{(k_2 - k_1)(k_4 - k_2)}{(k_3 - k_1)(k_4 - k_3)} - 2 \ln a = 0. \end{cases}$$

Then following the same procedure as in the case of O-type, we get the amplitude at the center of interaction which is located at the point of $x = y = t = 0$,

$$(6.14) \quad \begin{aligned} u(0, 0, 0) &= \frac{1}{2} ((k_4 - k_1)^2 + (k_3 - k_2)^2) - \frac{1 - \sqrt{\Delta_P}}{1 + \sqrt{\Delta_P}} (k_4 - k_1)(k_3 - k_2) \\ &= A_{[1,4]} + A_{[2,3]} - 2 \frac{1 - \sqrt{\Delta_P}}{1 + \sqrt{\Delta_P}} \sqrt{A_{[1,4]} A_{[2,3]}}, \end{aligned}$$

where Δ_P is given by

$$\Delta_P = \frac{(k_2 - k_1)(k_4 - k_3)}{(k_3 - k_1)(k_4 - k_2)}.$$

If we set $x = \frac{k_2 - k_1}{k_4 - k_1}$, $y = \frac{k_4 - k_3}{k_4 - k_1}$, then we have

$$\frac{1 - \sqrt{\Delta_P}}{1 + \sqrt{\Delta_P}} = \frac{\sqrt{(1-x)(1-y)} - \sqrt{xy}}{\sqrt{(1-x)(1-y)} + \sqrt{xy}} = \frac{1 - (x+y)}{(\sqrt{(1-x)(1-y)} + \sqrt{xy})^2}.$$

Using the inequality (equivalent to $\sqrt{xy} \leq \frac{1}{2}(x+y)$),

$$\sqrt{(1-x)(1-y)} + \sqrt{xy} \leq 1.$$

we have

$$\frac{1 - \sqrt{\Delta_P}}{1 + \sqrt{\Delta_P}} \geq 1 - (x+y) = \frac{k_3 - k_2}{k_4 - k_1} = \sqrt{\frac{A_{[2,3]}}{A_{[1,4]}}}.$$

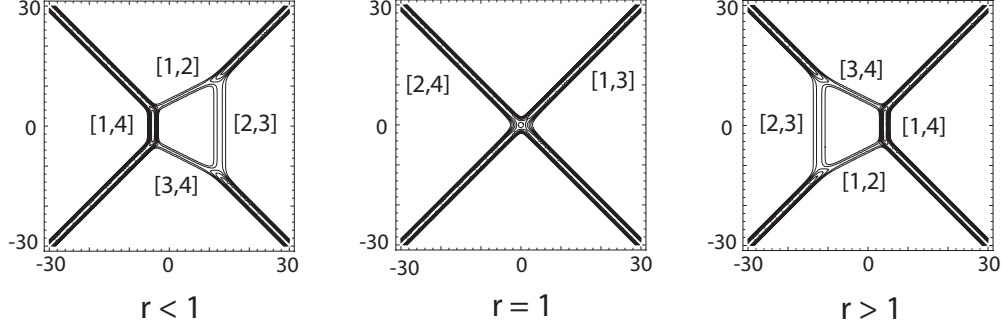


FIGURE 6.7. Two-soliton solution with T-type interaction. The parameters are chosen as $(k_1, k_2, k_3, k_4) = (-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2})$ and $s = 1$ (i.e. no phase shifts). Two other elements b, c in the A -matrix are chosen, so that two solitons intersect at the origin when $r = 1$. Also notice that T-type soliton consists of (2413)-soliton in $x < 0$ and (3142)-soliton in $x > 0$.

Then (6.14) gives the following bound of the interaction amplitude,

$$\left(\sqrt{A_{[1,4]}} - \sqrt{A_{[2,3]}} \right)^2 < u(0, 0, 0) \leq A_{[1,4]} - A_{[2,3]},$$

As a result, we find that the amplitude at the center of the amplitude is smaller than the larger soliton-amplitude $A_{[1,4]}$ due to the P-type interaction which acts as a repulsive force (see also the phase shift (6.13)).

6.4. T-type soliton solutions. There are four parameters in the A -matrix for T-type soliton solution consisting of two asymptotic line-solitons, $[1, 3]$ and $[2, 4]$. Here we explain that those parameters give the information of the locations of those line-solitons, the phase shift and on-set of the opening of a box (see Figure 6.7). Thus three of those four parameters are determined by the asymptotic data on large $|y|$, and we need an internal data for the other one.

Following the arguments in the previous section, one can find the phase shifts of the line-solitons of $[1, 3]$ and $[2, 4]$: For $[1, 3]$ -soliton in $x > 0$ (and $y \gg 0$), the phase shift is calculated as

$$\theta_{13}^+ = \ln \frac{k_4 - k_1}{k_4 - k_3} - \ln \frac{D}{b},$$

where $D = ad - bc = \xi(3, 4)$. For the same soliton in $x < 0$ (and $y \ll 0$), we have

$$\theta_{13}^- = \ln \frac{k_2 - k_1}{k_3 - k_2} - \ln c.$$

So the total phase shift is given by

$$\theta_{13} = \theta_{13}^+ - \theta_{13}^- = \ln \frac{(k_4 - k_1)(k_3 - k_2)}{(k_2 - k_1)(k_4 - k_3)} - \ln \frac{D}{bc}.$$

Thus the shift depends on the A -matrix unlike the cases of O- and P-types, and it can take any value.

For $[2, 4]$ -soliton in $x > 0$ (and $y \ll 0$), we have

$$\theta_{24}^+ = \ln \frac{k_3 - k_2}{k_4 - k_3} - \ln \frac{D}{c}.$$

and for the same one in $x < 0$ (and $y \gg 0$), we have

$$\theta_{24}^- = \ln \frac{k_2 - k_1}{k_4 - k_1} - \ln b.$$

Note that the total phase shift $\theta_{24} = \theta_{24}^+ - \theta_{24}^-$ is the same as that for $[1, 3]$ -soliton, i.e. the phase conservation along the y -axis $\theta_{13}^+ + \theta_{24}^- = \theta_{13}^- + \theta_{24}^+$ holds. Then as in (6.9) for the case of (3142)-type, we define the s -parameter,

$$(6.15) \quad s := \exp(-\theta_{13}^+ - \theta_{24}^-),$$

which represents the intersection point of $[1, 3]$ - and $[2, 4]$ -soliton. With the s -parameter, we have

$$(6.16) \quad b = \frac{k_2 - k_1}{k_4 - k_1} s e^{\theta_{13}^+}, \quad c = \frac{k_2 - k_1}{k_3 - k_2} s e^{\theta_{24}^+}, \quad D = \frac{k_2 - k_1}{k_4 - k_3} s.$$

Namely, the three parameters b, c and $D = ad - bc$ determine the locations and the phase shift (i.e. the intersection point of $[1, 3]$ - and $[2, 4]$ -solitons). One other parameter is then related to an on-set of a box at the intersection point (see Figure 6.7).

In order to characterize this parameter, let us consider the intermediate solitons of $[1, 4]$ and $[2, 3]$. First note that for $t \gg 0$, $[1, 4]$ -soliton appears as the dominant balance between $E(1, 2)$ and $E(2, 4)$. This implies that the τ -function can be written by

$$\tau \approx E(1, 2) + dE(2, 4) \equiv \cosh \frac{1}{2}(\theta_1 - \theta_4 + \theta_{14}^+),$$

where the super index “+” means $t > 0$. Then the phase shift θ_{14}^+ is given by

$$\theta_{14}^+ = \ln \frac{k_2 - k_1}{k_4 - k_2} - \ln d.$$

Similarly one can get the phase shift θ_{14}^- for $t \ll 0$ as

$$\theta_{14}^- = \ln \frac{k_3 - k_1}{k_4 - k_3} - \ln \frac{D}{a}.$$

Now consider the sum of θ_{14}^\pm , i.e.

$$\theta_{14}^+ + \theta_{14}^- = \ln \frac{(k_2 - k_1)(k_3 - k_1)}{(k_4 - k_2)(k_4 - k_3)} - \ln \frac{dD}{a}.$$

Also, for the $[2, 3]$ -soliton, one can get

$$\theta_{23}^+ + \theta_{23}^- = \ln \frac{(k_2 - k_1)(k_4 - k_2)}{(k_3 - k_1)(k_4 - k_3)} - \ln \frac{aD}{d}.$$

Now we introduce a parameter r in the form,

$$(6.17) \quad \frac{a}{d} = r \frac{k_4 - k_2}{k_3 - k_1},$$

so that we have

$$\theta_{14}^+ + \theta_{14}^- = \ln \frac{r}{s}, \quad \theta_{23}^+ + \theta_{23}^- = -\ln(rs).$$

Suppose that at $t = 0$, $[1, 3]$ - and $[2, 4]$ -solitons in $x > 0$ are placed so that they meet at the origin, that is, we choose $\theta_{13}^+ = \theta_{24}^+ = 0$. Also if there is no phase shifts for those solitons, i.e. $s = 1$. then the sums become

$$\theta_{14}^+ + \theta_{14}^- = \ln r = -(\theta_{23}^+ + \theta_{23}^-).$$

This implies that at $t = 0$ if $r = 1$, then the T-type soliton solution has an exact shape of “X” without any opening of a box at the intersection point on the origin. Moreover, at $t = 0$ if $r > 1$, then $[1, 4]$ -soliton appears in $x > 0$ and $[2, 3]$ -soliton in $x < 0$; whereas if $0 < r < 1$, then $[1, 4]$ -soliton appears in $x < 0$ and $[2, 3]$ -soliton in $x > 0$. Figure 6.7 illustrates those cases with $s = 1$. The parameter r determines the exponential term that is dominant in the region inside the box. When $r < 1$, $E(2, 4)$ is the dominant exponential term, and when $r > 1$ the dominant exponential is $E(1, 3)$. One should note that the parameter r cannot be determined by the asymptotic data, that is, r is considered as an “internal” parameter.

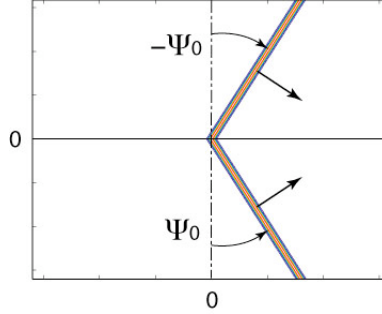


FIGURE 7.1. Initial data with V-shape wave. Each line of the V-shape is locally a line-soliton solution, $u(x, y, 0) = A_0 \operatorname{sech}^2 \sqrt{\frac{1}{2}A_0}(x + c_0 y)$ with $c_0 = \tan(\pm\Psi_0)$. We set those line-solitons to meet at the origin. In our numerical simulation, we reverse the time, i.e. $t \rightarrow -t$, so that the solitons are propagate in the positive x -direction, and small radiations propagate in the negative x -direction.

7. NUMERICAL SIMULATION WITH V-SHAPE INITIAL WAVES

In this section, we present some numerical simulations of the KP equation with obliquely incident initial waves related to a physical situation (see for examples [29, 33, 12]). In these simulations, we change the sign of the time t , so that now the solitons move in the positive x -direction (this seems to be a convention in the area of fluid mechanics). A complete numerical study of the initial value problem will be reported in a future communication [16].

We consider the initial data given in the shape of “V”,

$$(7.1) \quad u(x, y, 0) = A_0 \operatorname{sech}^2 \sqrt{\frac{1}{2}A_0}(x - c_0|y|),$$

where $c_0 = \tan \Psi_0 > 0$ with the angle measured from the y -axis (see Figure 7.1). Note here that two semi-infinite line-solitons are propagating toward each other, so that they interact strongly at the corner of the V-shape. At the boundaries $y = \pm y_{\max}$ of the numerical domain, those line-solitons are patched to the exact one-soliton solutions given by

$$u(x, \pm y_{\max}, t) = A_0 \operatorname{sech}^2 \sqrt{\frac{1}{2}A_0}(x \mp c_0 y_{\max} - \nu t),$$

with $\nu = \frac{3}{4}c_0^2 + \frac{1}{2}A_0$. Notice that these solitons correspond to the exact one-soliton solution of the KdV equation. The numerical simulations are based on a spectral method with window-technique similar to the method used in [33] (the details will be reported in [16]). The V-shape initial wave was first considered by M. Oikawa and H. Tsuji (see for example [29, 33]) in order to understand the generation of freak (or rogue) waves. They noticed generations of different types of asymptotic solutions, and found the resonant interactions which create localized high amplitude waves. In this paper, we present the results for the cases corresponding to $A_0 = 2$ and two different angles, Ψ_1 and Ψ_2 with $\Psi_1 < \Psi_c < \Psi_2$, where the critical angle is given by $\Psi_c = \tan^{-1} \sqrt{2A_0} \approx 63.4^\circ$. Then we explain these results in terms of certain $(2, 2)$ -soliton solutions discussed in the previous section, and in particular, we describe the connection with the Mach reflection discussed in [22].

The main idea here is to consider the V-shape initial wave as the part of some $(2, 2)$ -soliton solution listed in the previous section. In order to identify those soliton solutions from the V-shape initial wave form, let us first denote them as $[i_1, j_1]$ -soliton for $y \gg 0$ and $[i_2, j_2]$ -soliton for $y \ll 0$. Then using the relations, $k_j - k_i = \sqrt{2A_0}$ and $k_j + k_i = \tan \Psi_0$, for $[i, j]$ -soliton, we have

$$(7.2) \quad \begin{cases} k_{j_2} = \frac{1}{2}(\tan \Psi_0 + 2), & k_{i_2} = \frac{1}{2}(\tan \Psi_0 - 2), \\ k_{j_1} = -\frac{1}{2}(\tan \Psi_0 - 2), & k_{i_1} = -\frac{1}{2}(\tan \Psi_0 + 2). \end{cases}$$

Because of the symmetry in the initial shape, we also have $k_{j_2} = -k_{i_1}$ and $k_{i_2} = -k_{j_1}$. Moreover, at the critical angle, $\tan \Psi_c = \sqrt{2A_0} = 2$, we have $k_{i_2} = k_{j_1} = 0$. We also identify k_{i_1} as the smallest parameter and k_{j_2} as the largest one, so that depending on the angle Ψ_0 , we obtain the following ordering in the k -parameters:

For $0 < \Psi_0 < \Psi_c$ (i.e. $0 < \tan \Psi_0 < 2$), we have

$$k_{i_1} < k_{i_2} < 0 < k_{j_1} < k_{j_2},$$

implying that the corresponding chords of the $[i_1, j_1]$ - and the $[i_2, j_2]$ -solitons overlap. This means that the two solitons can be identified as part of either the (3412)-type (T-type) or the (3142)-type solution (see Figure 6.1). That is, $[1, 3]$ chord appears on the upper side of the diagram, and $[2, 4]$ chord on the lower side. As will be shown below, the numerical simulation suggests that the solution converges asymptotically to a (3142)-type soliton solution. This situation is related to the Mach reflection discussed in [22, 12].

For $\Psi_c < \Psi_0 < \frac{\pi}{2}$ (i.e. $2 < \tan \Psi_0$), we have

$$k_{i_1} < k_{j_1} < 0 < k_{i_2} < k_{j_2}.$$

In this case, the corresponding chords are separated implying that the two solitons form part of either (2413)- or (2143)-type (O-type) solution. That is, $[1, 2]$ - and $[3, 4]$ -chords appear on the upper and lower sides of the chord diagram, respectively. The numerical simulation seems to indicate that the solution converges asymptotically to an O-type solution, and this corresponds to the regular reflection [22, 12].

We can thus predict the following types of the asymptotic solutions depending on the values Ψ_0 :

- (a) If the angle is $\Psi_0 = 0$, then the initial wave is just one-soliton solution. This soliton is parallel to the y -axis, so $k_1 = -k_2$ showing no y -dependency.
- (b) If the angle is in the range $0 < \Psi_0 < \Psi_c = \tan^{-1} \sqrt{2A_0}$, then the asymptotic solution is the (3142)-type soliton solution (not T-type)
- (c) If the angle satisfy $\Psi_c < \Psi_0 < \frac{\pi}{2}$, then the asymptotic solution is the (2143)-type soliton solution (i.e. O-type, and not (2413)-type).

In the next two subsections, we present the results for the cases (b) and (c) (the case (a) is obvious).

7.1. Numerical simulation for $\Psi_0 > \Psi_c$. In Figure 7.2, the upper three figures illustrate the numerical simulations of the initial value problem for the V-shape initial wave with $A_0 = 2$ and $\Psi_0 = \tan^{-1}(3) \approx 71.57^\circ$. Here the critical angle is $\Psi_c = \tan^{-1}(2) \approx 63.4^\circ$, and we observe asymptotically an O-type soliton solution. The corresponding k -parameters are obtained from (7.2), i.e.

$$(k_1, k_2, k_3, k_4) = \left(-\frac{5}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{5}{2}\right).$$

Since two solitons $[1, 2]$ - and $[3, 4]$ -solitons for $x > 0$ in the initial data are $[1, 2]$ - and $[3, 4]$ -solitons, which meet at the origin, i.e. $\theta_{12}^+ = \theta_{34}^+ = 0$, we take the A -matrix in the form,

$$A = \begin{pmatrix} 1 & \frac{5}{3} & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{3} \end{pmatrix},$$

where we have used (6.1) and (6.2), i.e. $a = \frac{k_4 - k_1}{k_4 - k_2} = \frac{5}{3}$ and $b = \frac{k_4 - k_2}{k_4 - k_1} = \frac{1}{3}$. The corresponding O-type exact soliton solution is illustrated in the lower figures in Figure 7.2. We also confirm that the numerical solution is indeed converging to this exact solution (details of these results will be reported in [16]).

7.2. Numerical simulation for $\Psi_0 < \Psi_c$. In Figure 7.3, the upper figures illustrate the result of a simulation with the initial V-shape wave (7.1) with $A_0 = 2$ and $\Psi_0 = 45^\circ$. The angle Ψ_0 is now less than the critical angle Ψ_c . The asymptotic solution is expected to be of (3142)-type whose parameters are determined as follows:

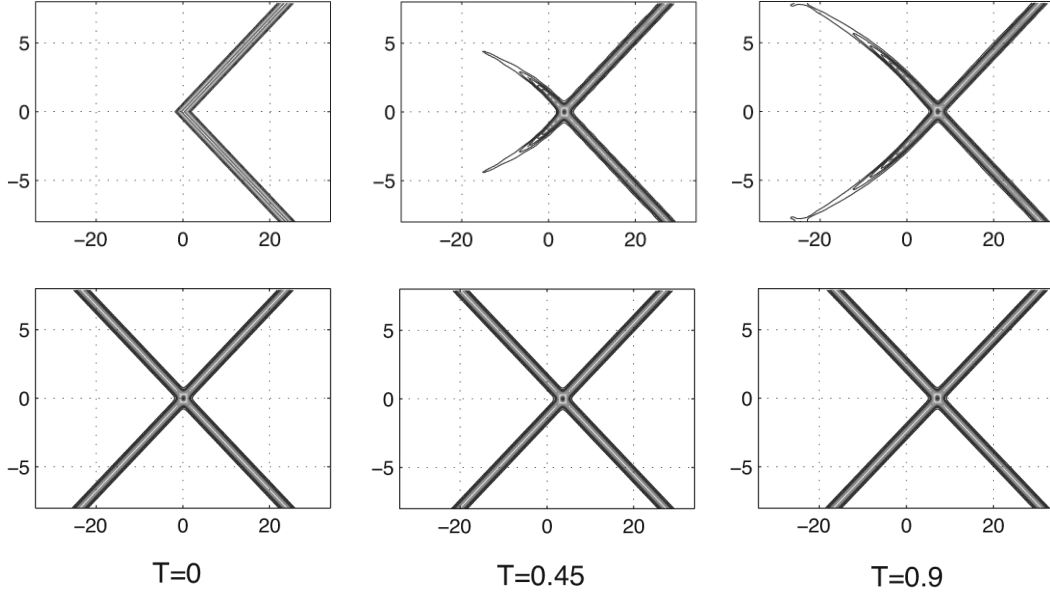


FIGURE 7.2. Numerical simulation with the V-shape initial wave and an exact solution corresponding to the asymptotic solution. The parameters are given by $A_0 = 2$ and $\Psi_0 = \tan^{-1}(3) = 71.6^\circ > \Psi_c = \tan^{-1}(2) \approx 63.4^\circ$. The solution converges asymptotically to the O-type solution shown in the three figures in the second line. The parameters are given by $(k_1, k_2, k_3, k_4) = (-\frac{5}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{5}{2})$.

- (i) The k -parameters can be obtained from (7.2) above,

$$(k_1, k_2, k_3, k_4) = (-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}).$$

The angles of the line-solitons of $[1, 2]$ - and $[3, 4]$ -types in $x \ll 0$ are then given by

$$\Psi_{[3,4]} = -\Psi_{[1,2]} = \Psi_c = \tan^{-1}(2) \approx 63.4^\circ.$$

- (ii) The elements of the A -matrix may be obtained by the condition that the intersection point at $t = 0$ is located at the origin for the initial solitons identified as $[1, 3]$ - and $[2, 4]$ -types. Then from the phase shift formulas (6.8), we choose $\frac{c}{b} = 3$ and $\frac{c}{a} = 1$, so that $[1, 3]$ - and $[2, 4]$ -solitons meet at the origin at $t = 0$. We also expect that all four solitons (two original in $x > 0$ and two other solitons in $x < 0$) may be considered to intersect at the origin at $t = 0$. This assumption leads to $s = 1$ in (6.10), and we obtain

$$A = \begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & \frac{2}{3} \end{pmatrix}.$$

However this choice of the s -parameter may not be correct, since the generation of the line-solitons in $x < 0$ affects the interaction timing at $t = 0$ (it seems that a correct choice is $s < 1$, and this will be discussed further in [16]).

The lower figures in Figure 7.3 illustrate the corresponding exact solution with this A -matrix. One can observe that the asymptotic solution with the V-shape initial wave with $\Psi_0 = 45^\circ < \Psi_c$ converges to the corresponding (3142)-type soliton solution. An explicit error analysis for the results will be also reported in [16].

7.3. The Mach reflection. In [22], J. Miles considered an oblique interaction of two line-solitons using O-type solutions. He observed that resonance occurs at the critical angle Ψ_c , and when the angle Ψ_0 is smaller than Ψ_c , the O-type solution becomes singular (recall that at the critical angle

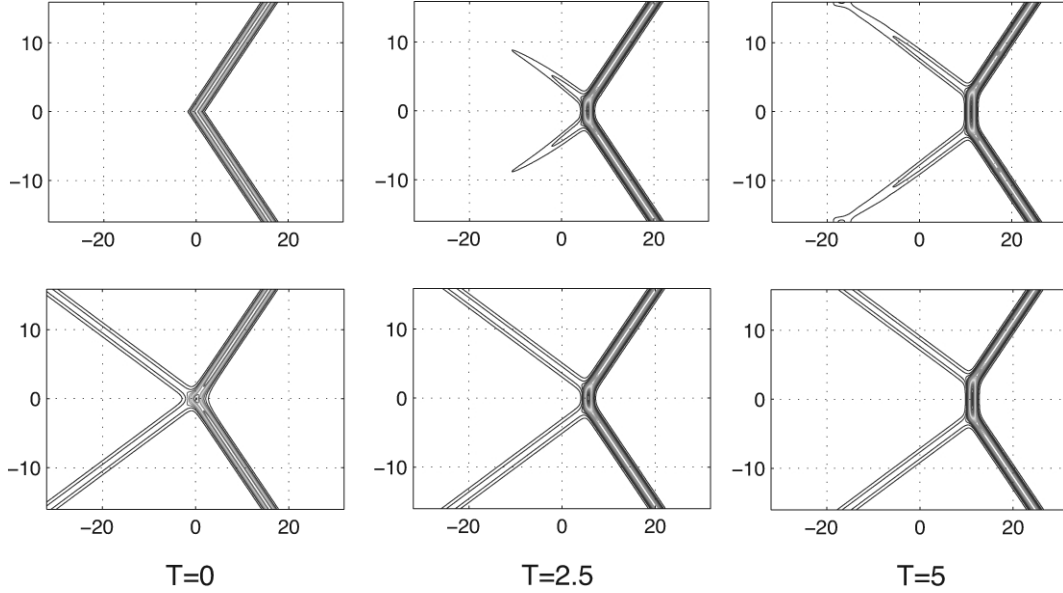


FIGURE 7.3. Numerical simulation with the V-shape initial wave and an exact solution corresponding to the asymptotic solution. The parameters are given by $A_0 = 2$ and $\Psi_0 = 45^\circ$. The intermediate wave has the amplitude about 4.5. This solution seems to converge asymptotically to the (3142)-type solution shown in the second line which is generated by the A -matrix with $a = 2, b = \frac{2}{3}$ and $c = 2$ (i.e. all four line-solitons meet at the origin at $t = 0$). The parameters are given by $(k_1, k_2, k_3, k_4) = (-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2})$.

Ψ_c , one of the exponential term in the τ -function vanishes). He also noticed a similarity between this resonant interaction and the Mach reflection found in a shock wave interaction. This may be illustrated by the left figure of Figure 7.4, where an incidence wave shown by the vertical line is propagating to the right, and it hits a rigid wall with the angle $-\Psi_0$ measured counterclockwise from the axis perpendicular to the wall (see also [12]). If the angle of the incidence wave (equivalently the inclination angle of the wall) is large, the reflected wave behind the incidence wave has the same angle Ψ_0 , i.e. a regular reflection occurs. However, if the angle is small, then an intermediate wave called the Mach stem appears as shown in Figure 7.4. The critical angle for the Mach reflection is given by the angle Ψ_c . The Mach stem, the incident wave and the reflected wave interact resonantly, and those three waves form a resonant triplet. The right one in Figure 7.4 illustrates an equivalent system of the wave propagation in the left figure (one should ignore the effect of viscosity on the wall, i.e. no boundary layer). At the point O , the initial wave has V-shape with the angle Ψ_0 , which forms the initial data for the simulation. Then the numerical simulation describes the reflection of line-soliton with an inclined wall, and these results explain well the appearance of the Mach reflection in terms of the exact soliton solution of (3142)-type.

The maximum amplitude for this problem occurs at the wall, called the maximum run-up, and this can be obtained by the formula (6.7) for the O-type solution, i.e. $\Psi_0 > \Psi_c$, and the formula (6.12) for (3142)-type solution. Figure 7.5 illustrates the maximum amplitude for various angles Ψ_0 . Notice that at the critical angle $\Psi_c \approx 63.4^\circ$, the amplitude takes four times higher than that of the incident wave (this figure was first obtained in [22]). One can also find the length of the Mach stem (i.e. the intermediate soliton of [1, 4]-type) from (3142)-soliton solution, that is, for the point (x, y)

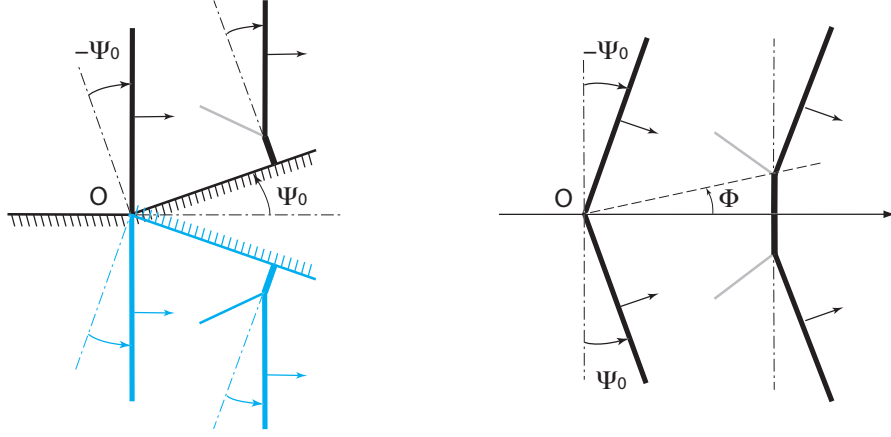


FIGURE 7.4. The Mach reflection. The left figure illustrates a semi-infinite line-soliton propagating parallel to the wall. The right figure is an equivalent system to the left one when we ignore the viscous effect on the wall. The resulting wave pattern is of (3142)-soliton solution.

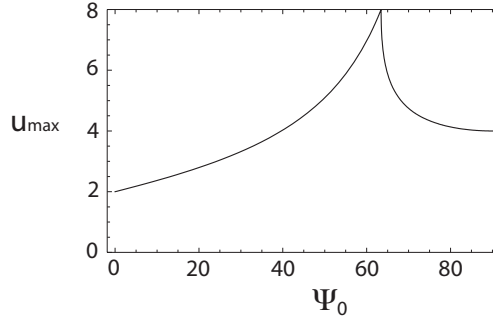


FIGURE 7.5. The maximum amplitude obtained from the O-type soliton for $\Psi_0 < \Psi_c$ and the (3142)-type soliton for $\Psi_0 > \Psi_c$. The critical angle is $\Psi_c = \tan^{-1} \sqrt{2A_0} \approx 63.4^\circ$. The highest amplitude is obtained at the critical angle, and it is 4 time larger than the initial soliton.

of the resonant interaction of the triplet, we have

$$x = \frac{1}{4} (\tan \Psi_c + \tan \Psi_0)^2 t, \quad y = \frac{1}{2} (\tan \Psi_c - \tan \Psi_0) t,$$

where $\tan \Psi_c = \sqrt{2A_0}$ and the point O is assigned as $(0, 0)$. The angle Φ in Figure 7.4 is then given by

$$\tan \Phi = \frac{2(\tan \Psi_c - \tan \Psi_0)}{(\tan \Psi_c + \tan \Psi_0)^2}.$$

This formula might be useful to determine the s -parameter for (3142)-type soliton solution.

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REFERENCES

- [1] M. J. Ablowitz and H. Segur, *Solitons and the inverse scattering transform*, SIAM Studies in Applied Mathematics, (SIAM, Philadelphia/ 1981).
- [2] M. J. Ablowitz and P. A. Clarkson, *Solitons, nonlinear evolution equations and inverse scattering* (Cambridge University Press, Cambridge, 1991)
- [3] G. Biondini and S. Chakravarty, Soliton solutions of the Kadomtsev-Petviashvili II equation, *J. Math. Phys.*, **47** (2006) 033514.
- [4] G. Biondini and Y. Kodama, On a family of solutions of the Kadomtsev-Petviashvili equation which also satisfy the Toda lattice hierarchy, *J. Phys. A: Math. Gen.* **36** (2003) 10519-10536.
- [5] M. Boiti, F. Pempinelli, A. K. Pogrebkov and B. Prinari, Towards an inverse scattering theory for non-decaying potentials of the heat equation, *Inverse Problems*, **17** (2001) 937-957.
- [6] S. Chakravarty and Y. Kodama, Classification of the line-solitons of KP II, *J. Phys. A: Math. Theor.*, **41** (2008) 275209 (33pp).
- [7] S. Chakravarty and Y. Kodama, A generating function for the N -soliton solutions of the Kadomtsev-Petviashvili II equation, *Contemp. Math.*, **471** (2008) 47-67.
- [8] L. A. Dickey, *Soliton equations and Hamiltonian systems*, Advanced Series in Mathematical Physics, Vol. **12**, (World Scientific, Singapore, 1991).
- [9] W-S. Duan, Y-R. Shi and X-R. Hong, Theoretical study of resonance of the Kadomtsev-Petviashvili equation, *Phys. Lett. A*, **323** (2004) 89-94.
- [10] P. A. Folkes, H. Ikezi and R. Davis, Two-dimensional interaction of ion-acoustic solitons, *Phys. Rev. Lett.* **45** (1980) 902-904
- [11] N. C. Freeman and J. J. C. Nimmo, Soliton-solutions of the Korteweg-deVries and Kadomtsev-Petviashvili equations: the Wronskian technique *Phys. Lett. A* **95** (1983) 1-3.
- [12] M. Funakoshi, Reflection of obliquely incident solitary waves, *J. Phys. Soc. Jpn*, **49**, (1980) 2371-2379.
- [13] H. Harada, New subhierarchies of the KP hierarchy in the Sato theory. 2. Truncation of the KP hierarchy, *J. Phys. Soc. Jpn*, **56** (1987) 3847-3852.
- [14] R Hirota, *The Direct Method in Soliton Theory* (Cambridge University Press, Cambridge, 2004)
- [15] B. B. Kadomtsev and V. I. Petviashvili, On the stability of solitary waves in weakly dispersive media, *Sov. Phys. - Dokl.* **15** (1970) 539-541.
- [16] C.-Y. Kao, and Y. Kodama, Numerical study on the initial value problem of the KP equation (in preparation).
- [17] Y. Kodama, Young diagrams and N -soliton solutions of the KP equation, *J. Phys. A: Math. Gen.*, **37** (2004) 11169-11190.
- [18] F. Kako and N. Yajima, Interaction of Ion-acoustic solitons in two-dimensional space, *J. Phys. Soc. Jpn*, **49** (1980) 2063-2071.
- [19] F. Kako and N. Yajima, Interaction of ion-acoustic solitons in multi-dimensional space. II, *J. Phys. Soc. Jpn*, **51** (1982) 311-322.
- [20] V. B. Matveev and M. A. Salle, *Darboux Transformations and Solitons* (Springer-Verlag, Berlin 1991).
- [21] E. Medina, An N soliton resonance for the KP equation: interaction with change of form and velocity, *Lett. Math. Phys.* **62** (2002) 91-99.
- [22] J. W. Miles, Resonantly interacting solitary waves, *J. Fluid Mech.*, **79** (1977) 171-179.
- [23] P. A. Millewski and J. B. Keller, Three dimensional surface waves, *Stud. Appl. Math.*, **37** (1996) 149-166.
- [24] T. Miwa and M. Jimbo and E. Date, *Solitons: differential equations, symmetries and infinite-dimensional algebras* (Cambridge University Press, Cambridge, 2000)
- [25] A. C. Newell, *Solitons in Mathematics and Physics*, CBMS-NSF **48**, (SIAM, Philadelphia/ 1985).
- [26] A. C. Newell and L. G. Redekopp, Breakdown of Zakharov-Shabat theory and soliton creation, *Phys. Rev. Lett.* **38** (1977) 377-380.
- [27] T. Nagasawa and Y. Nishida, Virtual states in strong interactions of plane ion-acoustic solitons, *Phys. Rev. A*, **28** (1983) 3043-3050.
- [28] S. Novikov, S. V. Manakov, L. P. Pitaevskii and V. E. Zakharov, *Theory of Solitons: The Inverse Scattering Method*, Contemporary Soviet Mathematics, (Consultants Bureau, New York and London, 1984).
- [29] A. V. Porubov, H. Tsuji, I. L. Lavrenov and M. Oikawa, Formation of the rogue wave due to non-linear two-dimensional waves interaction, *Wave Motion* **42** (2005) 202-210.
- [30] A. Postnikov, Total positivity, Grassmannians, and networks, (math.CO/0609764).
- [31] M. Sato, Soliton equations as dynamical systems on an infinite dimensional Grassmannian manifold, *RIMS Kokyuroku* (Kyoto University) **439** (1981) 30-46.

- [32] T. Soomere, Interaction of Kadomtsev-Petviashvili solitons with unequal amplitudes, *Phys. Lett. A*, **332** (2004) 74-78.
- [33] H. Tsuji and M. Oikawa, Oblique interaction of solitons in an extended Kadomtsev-Petviashvili equation, *J. Phys. Soc. Japan*, **76** (2007) 84401-84408.
- [34] G. B. Whitham, *Linear and nonlinear waves*, A Wiley-interscience publication (John Wiley & Sons, New York, 1974).
- [35] L. K. Williams, Enumeration of totally positive Grassmann cells, *Adv. Math.* **190** (2005) 319-342.
- [36] V. E. Zakharov and L. D. Fadeev, Korteweg-deVries equation, a completely integrable Hamiltonian system, *Funct. Anal. Appl.* **5** (1971) 280-287.

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